Mathematical Modelling and Analysis
Volume 28, Issue 3, 360–373, 2023
https://doi.org/10.3846/mma.2023.16620

Convergence of a Variational Iterative Algorithm for Nonlocal Vibrations Analysis of a Nanotube Conveying Fluid

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Received March 3, 2022; accepted June 12, 2023

Abstract. The amplitudes of the forced oscillations of a nano-structure conveying fluid are the solutions of an inhomogeneous integral-differential system. This is solved by an easily accessible scheme based on the variational iteration method (VIM), Galerkin's method and the Laplace transform techniques. The presented method is accompanied by the study of the convergence of the iterative process and of the errors. In the literature, the dynamic response of a viscoelastic nanotube conveying fluid is frequently obtained by an iterative method. This leads to the double convolution products, whose presence will be avoided in the new method proposed in this paper. Thus, the numerical results will be obtained much faster and more accurately.

Keywords: nanobeam conveying fluid, nonlocal calculus, Galerkin's method, variational iteration method, Laplace transform.

AMS Subject Classification: 44A10; 74A60; 74D10; 74G15; 74H15; 74H45.

1 Introduction

In the last decade, the interest for micro-nano scale engineering material has increased considerably in the modern science and technology. These structures possess small size, low density, high stiffness and thermal performances that are superior to the conventional materials. Both experimental and atomistic simulation results show that the nanoscale effect may not be neglected [5]. This is because at the small size the lattice spacing between the atoms becomes more important and discrete structure (internal) of the material can no longer be homogenized into a continuum. In the dynamic analysis of nanostructures, a special interest has the nonlocal continuum model, where the classical elasticity beam theories were augmented to incorporate the size-effects at small scale. The nonlocal theory takes into account the fact that stress at a reference point of the structure is affected not only by the strains at that point, but also by the strains at every point of it [7]. A review of the different nonlocal continuum models is presented in [2] and [15].

To establish the governing equations and the boundary conditions for the bending vibration of nanotubes, Lei et al. [9] proposed a Kelvin-Voigt viscoelastic model and the Timoshenko beam theory. The natural frequencies are computed with the transfer function methods (TFM). Using a variational differential quadrature (VDQ) method, Ansari et al. [1] analyzed the effects of the initial thermal loading on the vibrational behavior of embedded single-walled carbon nanotubes (SWCNTs) based on the nonlocal shell model. Several constitutive equations are presented in the paper [16] and a study on the nonlocal free vibration of the viscoelastic nanobeam with attached nanoparticles is presented by Cajic et al. [6]. The existing papers in the literature approach studies regarding only free vibrations of this structure, the material of the nanobeams being represented by a Kelvin-Voigt model, which has no instantaneous response in creep and shows unrealistic relaxation behavior. In recent years, the researchers are increasingly concerned with the dynamic analysis of the nanotubes conveying fluid, in which the sandwich structure is presented [18, 19]. The influences of the viscoelastic fractional Zener model parameters on the divergence and flutter flow velocity of the pipes conveying fluid are highlighted in [3, 4].

The aim of this paper is to present a method that avoids the multiple convolution products that appear today in the works dealing with the dynamic analysis of the micro-nano tubes conveying fluid [14]. A governing equation for a simply supported single walled carbon nanotube loaded to the uniformly distributed load is considered using a Euler-Bernoulli model. This equation corresponds to an improved viscoelastic model as the Zener model [10, 11], which is a linear combination of springs and dashpots. The proposed rheological model is accompanied by a constitutive law defined in an integral hereditary form [13, 17]. Written in terms of the transverse deflection of the nanobeam, the governing equation is solved with a Galerkin scheme. For the spatial domain are defined linearly independent, orthonormal shape functions, which satisfy the boundary conditions. Due to the presence of the fluid in the structure, the amplitudes of the oscillations become the solutions of an integral-differential system. This is solved using a Laplace variational iterative method (LVIM) [8], accompanied by a study on its convergence and errors. The validity of the method, which in this paper was applied for the first time to solving an integral-differential system, is demonstrated by a theorem. Using the computational efficiency of Laplace transform, the convolution theorem and algebraic calculations, the inverse of the integral transformation is found and finally, the solution of the system. The results of a numerical example are compared with the corresponding ones, which exist in literature for the same nanotube conveying fluid [14].

2 The governing equation

Consider a simply supported viscoelastic nanobeam of the length l for which a Zener rheological model was chosen. In accordance with [5], a nonlocal constitutive law defined by a hereditary integral is chosen

$$\bar{\sigma}(x,t) - \mu \frac{\partial^2 \bar{\sigma}(x,t)}{\partial x^2} = G(0) z \frac{\partial^2 w(x,t)}{\partial x^2} - \int_0^t \frac{dG(t-\tau)}{d\tau} z \frac{\partial^2 w(x,\tau)}{\partial x^2} d\tau,$$

where x and z represent the axial and the transverse coordinates, t is the time, $w(x,t)$ – the transversal deflection of the beam, $\bar{\sigma}(x,t)$ – the nonlocal stress and μ – the nonlocal parameter. The relaxation modulus has the following form

$$G(t) = k_2 \bar{k} \left(1 + \frac{k_1}{k_2} e^{-t/\tau_a}\right), \quad \tau_a = \frac{\eta}{k_1 + k_2}, \quad \bar{k} = \frac{k_1}{k_1 + k_2}, \tag{2.1}$$

where τ_a – the relaxation time, k_1, k_2 are the elastic modulus of the springs and η is the coefficient of viscosity of the dashpot. Using the d'Alembert's nonlocal principle is obtained the governing equation for the SWCNT conveying fluid [14]:

$$\rho S \frac{\partial^2 w(x,t)}{\partial t^2} - \mu \rho S \frac{\partial^4 w(x,t)}{\partial t^2 \partial x^2} + G(0) I \frac{\partial^4 w(x,t)}{\partial x^4} - I \int_0^t \frac{dG(t-\tau)}{d\tau} \frac{\partial^4 w(x,\tau)}{\partial x^4} d\tau$$
$$+ p_f(x,t) - m_f \left(2\nu\mu \frac{\partial^4 w(x,t)}{\partial x^3 \partial t} + \nu^2 \mu \frac{\partial^4 w(x,t)}{\partial x^4} + \mu \frac{\partial^4 w(x,t)}{\partial t^2 \partial x^2}\right) = \bar{p}, \tag{2.2}$$

where ρ is the density of the nanotube material, S – the area of the constant cross section S_t, $\bar{p}$ – the uniformly distributed load, I – the area moment of inertia defined by $\int\int_{S_t} z^2 dydz$, m_f and ν are the fluid mass per unit length of SWCNT and, respectively, the uniform mean flow velocity of fluid. The force per unit length induced by the fluid:

$$p_f(x,t) = m_f \left(2\nu \frac{\partial^2 w(x,t)}{\partial x \partial t} + \nu^2 \frac{\partial^2 w(x,t)}{\partial x^2} + \frac{\partial^2 w(x,t)}{\partial t^2}\right)$$

is the sum of three terms: the Coriolis, centrifugal and transverse forces.

3 Laplace variational iteration method for solving the integral-differential system

Applying the Galerkin's method, the solution of the Equation (2.2) is chosen of the form

$$w(x,t) = \sum_{i=1}^{n} a_i(t) \varphi_i(x), \tag{3.1}$$

where $a_i(t)$ is the corresponding time-dependent amplitude and $\varphi_i(x)$ is the ith shape function of the form

$$\varphi_i(x) = \sqrt{\frac{2}{l}} \sin \frac{i\pi x}{l}, \ \frac{d^4 \varphi_i(x)}{dx^4} = \left(\frac{i\pi}{l}\right)^4 \cdot \varphi_i(x) = \lambda_i \varphi_i(x), \ i=1,\dots,n. \tag{3.2}$$

The shape functions are chosen to be linearly independent, orthonormal and to satisfy the boundary conditions. If the sum (3.1) is substituted into (2.2), it will result the residual function $\bar{R}_n(x,t)$, [14]. Galerkin's method requires that the residual to be orthogonal to each of the shape functions, so

$$\int\int_{\Omega} \bar{R}_n(x,t)\varphi_j(x)dxdt = 0, \quad j = 1,2,\dots,n, \tag{3.3}$$

where $\Omega = [0,l] \times [0,t]$. The conditions (3.3) lead to n equations verified by the functions $a_j(t)$:

$$\begin{aligned}
&(m+m_f)\left(1+\mu\sqrt{\lambda_j}\right)\frac{d^2a_j(t)}{dt^2} \\
&\quad + 2\nu m_f \sum_{i=1}^{n}\left(\frac{da_j(t)}{dt}\int_0^l \frac{2\pi i}{l^2}\cos\left(\frac{\pi i x}{l}\right)\sin\left(\frac{\pi j x}{l}\right)dx\right)\left(1+\mu\sqrt{\lambda_i}\right) \\
&\quad - I\lambda_j\int_0^t \frac{dG(t-\tau)}{d\tau}a_j(\tau)d\tau + a_j(t)\left(G(0)I\lambda_j - \nu^2 m_f\sqrt{\lambda_j}(1+\mu\sqrt{\lambda_j})\right) \\
&= \bar{p}\int_0^l \varphi_j(x)dx,\ j=1,2,\dots,n,\ \text{where } m = \rho S.
\end{aligned} \tag{3.4}$$

The above equation will be divided with the product:$(m+m_f)(1+\mu\sqrt{\lambda_j})$ and the following notations are defined

$$\begin{aligned}
&\bar{\eta} = \frac{k_1 I}{m+m_f}, \quad \sigma = \frac{2m_f\nu}{m+m_f}, \quad b_j = \frac{\bar{\eta}\lambda_j}{1+\mu\sqrt{\lambda_j}} - \frac{m_f\nu^2\sqrt{\lambda_j}}{m+m_f}, \\
&P_j = \frac{\bar{p}}{(m+m_f)(1+\mu\sqrt{\lambda_j})}\int_0^l \varphi_j(x)dx = \frac{2\sqrt{2l}\bar{p}}{\pi j(m+m_f)(1+\mu\sqrt{\lambda_j})}, \\
&j - \text{odd number}, \\
&\bar{b}_j = \frac{\lambda_j\bar{\eta}\bar{k}}{1+\mu\sqrt{\lambda_j}} = \frac{k_1^2 I\lambda_j}{(m+m_f)(1+\mu\sqrt{\lambda_j})(k_1+k_2)}, \\
&q_j(t) = \frac{\bar{b}_j}{k_1\bar{k}}\int_0^t \frac{dG(t-\tau)}{d\tau}a_j(\tau)d\tau, \quad a_j'(t) = \frac{da_j(t)}{dt}.
\end{aligned} \tag{3.5}$$

From the form of P_j, it follows that as j increases, the importance of the $j-$th shape decreases. The numerical example will show that retaining the first four equations of the system a good approximation of the solution is obtained. Therefore, the system (3.4) becomes [14]:

$$\begin{aligned}
a_1''(t) + b_1a_1(t) + \alpha_1a_2'(t) + \alpha_2a_4'(t) - q_1(t) &= P_1, \\
a_2''(t) + b_2a_2(t) + \alpha_3a_1'(t) + \alpha_4a_3'(t) - q_2(t) &= 0, \\
a_3''(t) + b_3a_3(t) + \alpha_5a_2'(t) + \alpha_6a_4'(t) - q_3(t) &= P_3, \\
a_4''(t) + b_4a_4(t) + \alpha_7a_1'(t) + \alpha_8a_3'(t) - q_4(t) &= 0,
\end{aligned} \tag{3.6}$$

where

$$\alpha_1 = -\frac{8\sigma}{3l}\cdot\frac{1+\mu\sqrt{\lambda_2}}{1+\mu\sqrt{\lambda_1}}, \qquad \alpha_2 = -\frac{16\sigma}{15l}\cdot\frac{1+\mu\sqrt{\lambda_4}}{1+\mu\sqrt{\lambda_1}},$$

$$\alpha_3 = \frac{8\sigma}{3l}\cdot\frac{1+\mu\sqrt{\lambda_1}}{1+\mu\sqrt{\lambda_2}}, \qquad \alpha_4 = -\frac{24\sigma}{5l}\cdot\frac{1+\mu\sqrt{\lambda_3}}{1+\mu\sqrt{\lambda_2}},$$
$$\alpha_5 = \frac{24\sigma}{5l}\cdot\frac{1+\mu\sqrt{\lambda_2}}{1+\mu\sqrt{\lambda_3}}, \qquad \alpha_6 = -\frac{48\sigma}{7l}\cdot\frac{1+\mu\sqrt{\lambda_4}}{1+\mu\sqrt{\lambda_3}},$$
$$\alpha_7 = \frac{16\sigma}{15l}\cdot\frac{1+\mu\sqrt{\lambda_1}}{1+\mu\sqrt{\lambda_4}}, \qquad \alpha_8 = \frac{48\sigma}{7l}\cdot\frac{1+\mu\sqrt{\lambda_3}}{1+\mu\sqrt{\lambda_4}}.$$

Since the initial conditions on w and its derivatives are null, the following conditions for $a_j(t)$ are obtained

$$a_j(0) = 0, \qquad \left.\frac{da_j(t)}{dt}\right|_{t=0} = 0, \quad j = 1, 2, 3, 4.$$

The existence of a mixed derivative $\partial^2 w/\partial x\partial t$ in $p_f(x,t)$ leads to the dynamic coupling between the functions $a_j(t)$ of the modal solution. In the absence of the fluid, due to the orthogonality of the functions $\varphi_j(x)$, the unknowns $a_j(t)$ are determined independently of each other for $j = 1, 2, \ldots, n$, [13]. The matrix form of the system (3.6) is

$$A''(t) + BA(t) + \alpha A'(t) - Q(t) = P, \tag{3.7}$$

where $A(t) = [a_1(t)a_2(t)a_3(t)a_4(t)]^T$,

$$B = \begin{bmatrix} b_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & b_4 \end{bmatrix}, \quad \alpha = \begin{bmatrix} 0 & \alpha_1 & 0 & \alpha_2 \\ \alpha_3 & 0 & \alpha_4 & 0 \\ 0 & \alpha_5 & 0 & \alpha_6 \\ \alpha_7 & 0 & \alpha_8 & 0 \end{bmatrix},$$
$$Q(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \end{bmatrix}, \quad P = \begin{bmatrix} P_1 \\ 0 \\ P_3 \\ 0 \end{bmatrix}.$$

To use VIM will be written (3.7) as the sum of two operators:

$$\bar{L}A(t) + \bar{N}A(t) - P = 0,$$

where

$$\bar{L}A(t) = A''(t) + BA(t), \quad \bar{N}A(t) = \alpha A'(t) - \frac{\bar{b}_j}{k_1\bar{k}}\int_0^t \frac{dG(t-\tau)}{d\tau}A(\tau)d\tau.$$

The correction functional in t direction will be defined as:

$$A_{m+1}(t) = A_m(t) + \int_0^t \Lambda(\xi)(\bar{L}A_m(\xi) + \bar{N}\tilde{A}_m(\xi) - P)d\xi, \; m = 0, 1, 2, \ldots, \tag{3.8}$$

where the diagonal matrix Λ with elements $\tilde{\lambda}_i$, $i = 1, 2, 3, 4$, is the general Lagrange multiplier. The matrix function $\tilde{A}_m$ has a restricted variation in t

direction, which means $\delta\tilde{A}_m = 0$ and m denotes the $m-$th approximation. The integral can represent as a convolution, if the function Λ is of the following form

$$\Lambda(\xi) = \bar{\Lambda}(t-\xi).$$

Applying the Laplace transformation $\mathcal{L}$ in both members of the Equation (3.8), it is found that

$$\begin{aligned} \mathcal{L}(A_{m+1}(t)) &= \mathcal{L}(A_m(t)) + \mathcal{L}[\bar{\Lambda}(t) * (\bar{L}A_m(t) + \bar{N}\tilde{A}_m(\xi) - P)] \\ &= \mathcal{L}(A_m(t)) + \mathcal{L}(\bar{\Lambda}(t))\mathcal{L}(\bar{L}A_m(t) + \bar{N}\tilde{A}_m(t) - P). \end{aligned} \tag{3.9}$$

To obtain the approximation A_{m+1}, it will first be determined the new multiplier $\bar{\Lambda}$ using the variational techniques with respect to $A_m(t)$, [8]:

$$\begin{aligned} \mathcal{L}(\delta A_{m+1}(t)) &= \mathcal{L}(\delta A_m(t)) + \delta(\mathcal{L}(\bar{\Lambda}(t))\mathcal{L}(\bar{L}A_m(t))) \\ &= \mathcal{L}(\delta A_m(t)) + \mathcal{L}(\bar{\Lambda}(t))(s^2 I + B)\mathcal{L}(\delta A_m(t)) = 0, \end{aligned}$$

where I is the unity matrix. So, the above stationary condition leads to

$$\mathcal{L}(\bar{\Lambda}(t)) = -(s^2 I + B)^{-1},$$

where the four diagonal elements are

$$\mathcal{L}(\bar{\lambda}_i) = -1/(s^2 + b_i), \quad i = 1, 2, 3, 4.$$

Therefore, $\bar{\lambda}_i(t) = -\frac{1}{\sqrt{b_i}}\sin(t\sqrt{b_i})$ and $\tilde{\lambda}_i(\xi) = -\frac{1}{\sqrt{b_i}}\sin((t-\xi)\sqrt{b_i})$, $i = 1, 2, 3, 4$. Using (2.1) and (3.5) and a change of variable: $\theta(\tau) = t - \tau$ is obtained, [14]:

$$q_j(t) = -\frac{\bar{b}_j}{k_1\bar{k}}\int_0^t \frac{dG(\theta)}{d\theta} a_j(t-\theta)d\theta, \quad j = \overline{1,4}.$$

Recalling the formula: $\frac{df(t)}{dt} \xrightarrow{L} sF(s) - f(0)$, where $F(s)$ – the Laplace transform of $f(t)$, is found that

$$\mathcal{L}\left(\frac{dG(\theta)}{d\theta}\right) = \frac{-k_1\bar{k}}{1+\tau_a s}$$

and the form of the Laplace transform of the function $q_j(t)$ will be

$$q_j(t) \xrightarrow{L} Q_j(s) = -\frac{\bar{b}_j}{k_1\bar{k}}\mathcal{L}\left(\frac{dG(\theta)}{d\theta}\right)\mathcal{L}(A_j(t)) = \frac{\bar{b}_j}{1+\tau_a s}\mathcal{L}(A_j(t)), \quad j = \overline{1,4}. \tag{3.10}$$

Inserting (3.10) and $\mathcal{L}(\bar{\lambda}_i)$ in the iterative formula (3.9), it can be written

$$\begin{aligned} &\mathcal{L}(A_{m+1}(t)) = \mathcal{L}(A_m(t)) + \mathcal{L}(\bar{\Lambda}(t))\mathcal{L}(\bar{L}A_m(t) + \bar{N}\tilde{A}_m(t) - P(t)) \\ &= \mathcal{L}(A_m(t)) - (s^2 I + B)^{-1}\left[\left(s^2 I + B + s\alpha - \frac{\bar{B}}{1+\tau_a s}\right)\mathcal{L}(A_m(t)) - \frac{P}{s}\right] \\ &= (s^2 I + B)^{-1}\left[\left(-s\alpha + \frac{\bar{B}}{1+\tau_a s}\right)\mathcal{L}(A_m(t)) + \frac{P}{s}\right], \end{aligned}$$

where $\bar{B}$ is the diagonal matrix with the elements $\bar{b}_i$. Finally,

$$\mathcal{L}\begin{bmatrix} a_1^{(m+1)}(t) \\ a_2^{(m+1)}(t) \\ a_3^{(m+1)}(t) \\ a_4^{(m+1)}(t) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\bar{b}_1}{(s^2+b_1)(1+\tau_a s)} & \frac{-s\alpha_1}{s^2+b_1} & 0 & \frac{-s\alpha_2}{s^2+b_1} \\ \frac{-s\alpha_3}{s^2+b_2} & \frac{\bar{b}_2}{(s^2+b_2)(1+\tau_a s)} & \frac{-s\alpha_4}{s^2+b_2} & 0 \\ 0 & \frac{-s\alpha_5}{s^2+b_3} & \frac{\bar{b}_3}{(s^2+b_3)(1+\tau_a s)} & \frac{-s\alpha_6}{s^2+b_3} \\ \frac{-s\alpha_7}{s^2+b_4} & 0 & \frac{-s\alpha_8}{s^2+b_4} & \frac{\bar{b}_4}{(s^2+b_4)(1+\tau_a s)} \end{bmatrix}$$

$$\times \mathcal{L}\begin{bmatrix} a_1^{(m)}(t) \\ a_2^{(m)}(t) \\ a_3^{(m)}(t) \\ a_4^{(m)}(t) \end{bmatrix} + \begin{bmatrix} \dfrac{P_1}{s(s^2+b_1)} \\ 0 \\ \dfrac{P_3}{s(s^2+b_3)} \\ 0 \end{bmatrix}$$

or in the matrix form

$$\mathcal{L}(A_{m+1}(t)) = \tilde{F}(s)\mathcal{L}(A_m(t)) + \tilde{P}(s). \tag{3.11}$$

If noted: $X_{m+1}(s) = \mathcal{L}(A_{m+1}(t))$, (3.11) can be

$$X_{m+1}(s) = \tilde{F}(s)X_m(s) + \tilde{P}(s). \tag{3.12}$$

4 Convergence of the iterative method

A sufficient condition for the convergence of the sequence $(X_{m+1}(s))_m$ to the unique solution $X(s) = \mathcal{L}(A(t))$ is that the norm of the matrix $\tilde{F}(s)$ is less that one: $\left\|\tilde{F}(s)\right\| < 1$. The calculation of the above norm requires establishing the interval of variation of the complex variable module.

Let M be the set of complex numbers: $M = \left\{s \in \mathbf{C} \mid \arg(s) \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]\right\}$. If $s = s_1 + is_2 \in M$, then

$$|s^2 + b_i| = \sqrt{(s_1^2+s_2^2)^2 + 2(s_1^2 - s_2^2)b_i + b_i^2} \geq \sqrt{|s|^4 + b_i^2} \geq \sqrt{2b_i}|s|.$$

If the elements of the matrix $\tilde{F}(s)$ are F_{ij}, the definition of norm is, [12]:

$$\left\|\tilde{F}(s)\right\| = \max_i \sum_{j=1}^{4} |F_{ij}|. \tag{4.1}$$

Let k be the line of the matrix that leads to the calculation of the norm. Using

(4.1), on the one hand is obtained:

$$\left\|\tilde{F}(s)\right\| = \left(\left|\frac{\bar{b}_k}{1+\tau_a s}\right| + (\alpha_{2k-1}+\alpha_{2k})|s|\right)\frac{1}{|s^2+b_k|}$$
$$\leq \frac{\bar{b}_k}{\tau_a |s|^3} + \frac{(\alpha_{2k-1}+\alpha_{2k})|s|}{\sqrt{2b_k}|s|} \leq 1 \Rightarrow |s| \geq \sqrt[3]{\bar{b}_k/(0.9\tau_a)},$$

where, according to the numerical example

$$(\alpha_{2k-1}+\alpha_{2k})/\sqrt{2b_k} \leq 0.1.$$

On the other hand, it is found that

$$\left\|\tilde{F}(s)\right\| \leq \frac{1}{b_k}\left(\frac{\bar{b}_k}{\tau_a|s|} + (\alpha_{2k-1}+\alpha_{2k})|s|\right) \leq \frac{|s|}{b_k}\left(\frac{\bar{b}_k}{\tau_a|s|^2} + \alpha_{2k-1}+\alpha_{2k}\right)$$
$$\Rightarrow \frac{|s|}{b_k}\left(\frac{\bar{b}_k}{\tau_a\sqrt[3]{\left(\bar{b}_k/(0.9\tau_a)\right)^2}} + \alpha_{2k-1}+\alpha_{2k}\right) = |s|\bar{\beta} < 1 \Rightarrow |s| < 1/\bar{\beta},$$

where β is a value calculated in the Section 7. Let $|s| = 0.97/\beta$, so

$$\left\|\tilde{F}(s)\right\| = \bar{\beta}\cdot 0.97/\bar{\beta} = 0.97.$$

Therefore, the applicability of the presented method corresponds to the domain:

$$\sqrt[3]{\bar{b}_k/(0.9\tau_a)} \leq |s| < \frac{1}{\bar{\beta}},\ \arg(s) \in \left[-\frac{\pi}{4},\frac{\pi}{4}\right].$$

Returning to the iterative formula (3.12) and considering $X_0(s) = 0$, it will be obtained successively

$$X_1(s) = \tilde{P}(s),\ X_2(s) = (\tilde{F}(s)+I)\tilde{P}(s),\ X_3(s) = (\tilde{F}^2(s)+\tilde{F}(s)+I)\tilde{P}(s),\ldots.$$

Let $X_m(s)$ and $X_{m+1}(s)$ be two successive approximations of the solution $X(s)$. Then,

$$\|X_{m+1}(s) - X_m(s)\| = \left\|\tilde{F}^m(s)\right\|\,\left\|\tilde{P}(s)\right\|.$$

For $p \geq 1$, $p \in \mathbf{N}$, it is found that

$$\|X_{m+p}(s) - X_m(s)\| = \left\|\tilde{F}^{m+p-1}(s) + \tilde{F}^{m+p-2}(s) + \ldots + \tilde{F}^m(s)\right\|\,\left\|\tilde{P}(s)\right\|.$$

Finally, passing to the limit for $p \to \infty$, the error made for $X(s) \cong X_m(s)$ will be of the form

$$\varepsilon(s) = \|X(s) - X_m(s)\| = \frac{\left\|\tilde{F}(s)\right\|^m}{1-\left\|\tilde{F}(s)\right\|}\left\|\tilde{P}(s)\right\|.$$

5 Solving of the time-domain system

To find the solution of the matrix equation (3.7), the initial matrix is considered as: $A_0(t) = [0\,0\,0\,0]^T$. According to the iterative formula (3.11) results

$$A_1(t) = \mathcal{L}^{-1}(\tilde{P}(s)) = \left[\frac{P_1(1-\cos(\sqrt{b_1}t))}{b_1}\ 0\ \frac{P_3(1-\cos(\sqrt{b_3}t))}{b_3}\ 0\right]^T = \pi(t).$$

In the next iterative step, it is found that

$$\mathcal{L}(A_2(t)) = \tilde{F}(s)\mathcal{L}(A_1(t)) + \tilde{P}(s). \tag{5.1}$$

If the elements of the matrix $\tilde{P}(s)$ are $\tilde{p}_i(s)$, $i = 1,2,3,4$, and the elements of the matrix $\pi(t)$ are $\pi_i(t)$, then the relation (5.1) can be written as

$$\mathcal{L}\begin{bmatrix} a_1^{(2)}(t) \\ a_2^{(2)}(t) \\ a_3^{(2)}(t) \\ a_4^{(2)}(t) \end{bmatrix} = \begin{bmatrix} F_{11}(s)\tilde{p}_1(s) \\ F_{21}(s)\tilde{p}_1(s) + F_{23}(s)\tilde{p}_3(s) \\ F_{33}(s)\tilde{p}_3(s) \\ F_{41}(s)\tilde{p}_1(s) + F_{43}(s)\tilde{p}_3(s) \end{bmatrix} + \begin{bmatrix} \tilde{p}_1(s) \\ 0 \\ \tilde{p}_3(s) \\ 0 \end{bmatrix}.$$

The elements $F_{11}(s), F_{33}(s), \tilde{p}_1(s)$ and $\tilde{p}_3(s)$ break down into the simple fractions and it is obtained

$$\mathcal{L}^{-1}(F_{ii}(s)) = f_{i,i}(t) = \bar{b}_i\left(\beta_i\cos(\sqrt{b_i}t) + \gamma_i\sin(\sqrt{b_i}t) + \omega_i e^{-\frac{t}{\tau_a}}\right),\ i = 1,3,$$

where

$$\beta_i = -\tau_a/(1+b_i\tau_a^2),\quad \gamma_i = 1/\sqrt{b_i}(1+b_i\tau_a^2),\quad \omega_i = \tau_a/(1+b_i\tau_a^2).$$

To find the original function $a_i^{(2)}(t)$, $i = 1,2,3,4$ will use the Convolution theorem [5]:

$$a_{2k-1}^{(2)}(t) = \int_0^t f_{2k-1,2k-1}(t-\tau)\pi_{2k-1}(\tau)d\tau + \pi_{2k-1}(t),\quad k = 1,2,$$

$$a_2^{(2)}(t) = -\alpha_3\int_0^t \cos(\sqrt{b_2}(t-\tau))\pi_1(\tau)d\tau - \alpha_4\int_0^t \cos(\sqrt{b_2}(t-\tau))\pi_3(\tau)d\tau,$$

$$a_4^{(2)}(t) = -\alpha_7\int_0^t \cos(\sqrt{b_4}(t-\tau))\pi_1(\tau)d\tau - \alpha_8\int_0^t \cos(\sqrt{b_4}(t-\tau))\pi_3(\tau)d\tau.$$

The values of the temporal functions $a_j^{(m)}(t)$, $j = \overline{1,4}$, $m = 2,3,\ldots$ are presented below:

$$a_1^{(m+1)}(t) = \int_0^t f_{1,1}(t-\tau)a_1^{(m)}(\tau)d\tau + \pi_1(t) - \alpha_1\int_0^t \cos(\sqrt{b_1}(t-\tau))a_2^{(m)}(\tau)d\tau$$
$$- \alpha_2\int_0^t \cos(\sqrt{b_1}(t-\tau))a_4^{(m)}(\tau)d\tau,$$

$$a_3^{(m+1)}(t)=\int_0^t f_{3,3}(t-\tau)a_3^{(m)}(\tau)d\tau+\pi_3(t)-\alpha_5\int_0^t\cos(\sqrt{b_3}(t-\tau))a_2^{(m)}(\tau)d\tau$$
$$-\alpha_6\int_0^t\cos(\sqrt{b_3}(t-\tau))a_4^{(m)}(\tau)d\tau,$$
$$a_2^{(m+1)}(t)=\int_0^t f_{2,2}(t-\tau)a_2^{(m)}(\tau)d\tau-\alpha_3\int_0^t\cos(\sqrt{b_2}(t-\tau))a_1^{(m)}(\tau)d\tau$$
$$-\alpha_4\int_0^t\cos(\sqrt{b_2}(t-\tau))a_3^{(m)}(\tau)d\tau,$$
$$a_4^{(m+1)}(t)=\int_0^t f_{4,4}(t-\tau)a_4^{(m)}(\tau)d\tau-\alpha_7\int_0^t\cos(\sqrt{b_4}(t-\tau))a_1^{(m)}(\tau)d\tau$$
$$-\alpha_8\int_0^t\cos(\sqrt{b_4}(t-\tau))a_3^{(m)}(\tau)d\tau. \tag{5.2}$$

The maximum transverse displacement of the simply supported structure at the time t and $x=l/2$ will be determined with a high precision using (3.1)–(3.2) as follows

$$w(l/2,t)=\sqrt{\frac{2}{l}}(a_1^{(m)}(t)-a_3^{(m)}(t)).$$

6 Validation of the convergence of the iterative method

Let us now consider that the sequences $\left\{a_j^{(m)}(t)\right\}_m$, converge to the solution $a_j(t)$, $j=\overline{1,4}$ of the system (3.6). Therefore, the iterative process ceases if $a_j^{(m+1)}(t)=a_j^{(m)}(t)$ with an error less than a value $\tilde{\varepsilon}$ very small and the solution $a_j(t)$ will be equal to

$$a_j(t)=a_j^{(m+1)}(t)=a_j^{(m)}(t),\quad j=1,2,3,4.$$

Theorem 1. *If the functions* $a_j:D\to\mathbf{R}$, $D=[0,T]$, $a_j\in C^{(2)}(D)$, $j=\overline{1,4}$ *verifies the integral relations (5.2), where* $a_j(t)=a_j^{(m)}(t)=a_j^{(m+1)}(t)$, $j=1,2,3,4$, *then* $[a_1(t)a_2(t)a_3(t)a_4(t)]^T$ *is the solution of the integral-differential system (3.6).*

According to the hypothesis, the first relation in (5.2) will be of the form:

$$a_1(t)=\int_0^t f_{1,1}(t-\tau)a_1(\tau)d\tau-\alpha_1\int_0^t\cos(\sqrt{b_1}(t-\tau))a_2(\tau)d\tau$$
$$-\alpha_2\int_0^t\cos(\sqrt{b_1}(t-\tau))a_4(\tau)d\tau+\pi_1(t)$$

with

$$f_{1,1}(t)=\bar{b}_1\left(\frac{-\tau_a}{1+b_1\tau_a^2}\cos(\sqrt{b_1}t)+\frac{1}{\sqrt{b_1}(1+b_1\tau_a^2)}\sin(\sqrt{b_1}t)+\frac{\tau_a}{1+b_1\tau_a^2}e^{-\frac{t}{\tau_a}}\right)$$

and $f_{1,1}(0)=0$. It will be shown that $a_1(t)$ verifies the first equation of the system (3.6). Its derivatives are

$$a_1'(t)=\int_0^t \frac{\bar{b}_1}{1+b_1\tau_a^2}\left(\tau_a\sqrt{b_1}\sin(\sqrt{b_1}(t-\tau))+\cos(\sqrt{b_1}(t-\tau))-e^{-\frac{t-\tau}{\tau_a}}\right)$$
$$\times a_1(\tau)d\tau+f_{1,1}(0)a_1(t)+\alpha_1\int_0^t\sqrt{b_1}(\alpha_1a_2(\tau)+\alpha_2a_4(\tau))\sin(\sqrt{b_1}(t-\tau))d\tau$$
$$-\alpha_1a_2(t)-\alpha_2a_4(t)+\frac{P_1\sin(\sqrt{b_1}t)}{\sqrt{b_1}},$$
$$a_1''(t)=\int_0^t \frac{\bar{b}_1}{1+b_1\tau_a^2}\left(\tau_ab_1\cos\left(\sqrt{b_1}(t-\tau)\right)-\sqrt{b_1}\sin\left(\sqrt{b_1}(t-\tau)\right)+\frac{e^{-\frac{t-\tau}{\tau_a}}}{\tau_a}\right)$$
$$\times a_1(\tau)d\tau+\int_0^t b_1(\alpha_1a_2(\tau)+\alpha_2a_4(\tau))\cos(\sqrt{b_1}(t-\tau))d\tau$$
$$-\alpha_1a_2'(t)-\alpha_2a_4'(t)+P_1\cos(\sqrt{b_1}t).$$

From (2.1) and (3.5):

$$q_1(t)=\frac{\bar{b}_1}{k_1\bar{k}}\int_0^t\frac{dG(t-\tau)}{d\tau}a_1(\tau)d\tau=\frac{\bar{b}_1}{k_1\bar{k}}\int_0^t k_2\bar{k}\frac{k_1}{k_2\tau_a}e^{-\frac{t-\tau}{\tau_a}}a_1(\tau)d\tau$$
$$=\frac{\bar{b}_1}{\tau_a}\int_0^t e^{-\frac{t-\tau}{\tau_a}}a_1(\tau)d\tau.$$

Therefore, the first equation of the system (3.6) becomes:

$$a_1''(t)+b_1a_1(t)+\alpha_1a_2'(t)+\alpha_2a_4'(t)-q_1(t)$$
$$=\int_0^t\frac{\bar{b}_1}{1+b_1\tau_a^2}\left((\tau_ab_1-\tau_ab_1)\cos(\sqrt{b_1}(t-\tau))+(-\sqrt{b_1}+\sqrt{b_1})\sin(\sqrt{b_1}(t-\tau))\right.$$
$$\left.+e^{-\frac{t-\tau}{\tau_a}}\left(\frac{1}{\tau_a}+b_1\tau_a\right)\right)a_1(\tau)d\tau-\int_0^t\frac{\bar{b}_1e^{-\frac{t-\tau}{\tau_a}}}{\tau_a}a_1(\tau)d\tau-\alpha_1a_2'(t)-\alpha_2a_4'(t)$$
$$+\alpha_1a_2'(t)+\alpha_2a_4'(t)+P_1\cos(\sqrt{b_1}t)+b_1\frac{P_1(1-\cos(\sqrt{b_1}t))}{b_1}=P_1.$$

Proceeding analogously, it is proved that $a_2(t), a_3(t), a_4(t)$, defined by (5.2), also check the following three equations of the system (3.6).

7 Nonlocal dynamic model

In order to compare the results obtained with the iterative method used for the dynamic analysis of the fluid conveying nanotubes with those obtained in the paper [14], by another iterative algorithm, will be considered a structure with the same physical-mechanical properties and the same load. The simply supported beam is subjected to a uniformly distributed load $\bar{p}=5\cdot10^{-11}N/mm$. The beam has the length $l=25$ nm, the outer diameter $D=3.6$ nm, the thickness of nanotube $h=0.25$ nm and the inertial moment $I=3.7nm^4$ The following values of the density are adopted: $\rho=2300\cdot10^{-36}Ns^2/nm^4$ for nanobeam and $\rho_f = 1000 \cdot 10^{-36} Ns^2/nm^4$ for the fluid. So, the corresponding masses will have the values: $m = 6 \cdot 10^{-33} Ns^2/nm^2$ and $m_f = 7.5 \cdot 10^{-33} Ns^2/nm^4$. The uniform mean velocity of the flow fluid is chosen as $\nu = 5 \cdot 10^{11} nm/s$. The rheological model will be a nonlocal Zener model for which

$$k_1 = 3.4 \cdot \frac{10^{-6} N}{nm^2}, \qquad k_2 = 8.5 \cdot 10^{-7} \frac{N}{nm^2}, \qquad \eta = 0.15 \cdot 10^{-15} \frac{N \cdot s}{nm^2},$$
$$\tau_2 = 1.765 \cdot 10^{-10} s, \qquad \tau_a = 3.53 \cdot 10^{-11} s.$$

The above data leads to the following matrices:

$$\alpha = \begin{bmatrix} 0 & 6 & 0 & 3 \\ 6 & 0 & 12 & 0 \\ 0 & 10 & 0 & 17 \\ 2 & 0 & 14 & 0 \end{bmatrix} \cdot 10^{10}, \quad P = \begin{bmatrix} 16.3 \\ 0 \\ 4.8 \\ 0 \end{bmatrix} \cdot 10^{21}$$

and $b_3 = 164 \cdot 10^{23}$, $\bar{b}_3 = 131 \cdot 10^{23}$,

$$\frac{(\alpha_{2k-1} + \alpha_{2k})}{\sqrt{2b_k}} = \max\{0.1, 0.07, 0.05, 0.02\} = 0.1 \Rightarrow 7.5 \cdot 10^{11} \le |s| \le 1.7 \cdot 10^{13}.$$

To estimate the error corresponding to the iteration step $m = 3$ and $|s| = 7.5 \cdot 10^{11}$, the following norm is calculated $\left\| \tilde{P}(s) \right\| = 4 \cdot 10^{-14}$ and the norm of the error becomes

$$\varepsilon(s) = \|X(s) - X_3(s)\| = \frac{\left\|\tilde{F}(s)\right\|^3}{1 - \left\|\tilde{F}(s)\right\|} \left\| \tilde{P}(s) \right\| = \frac{0.97^3}{1 - 0.97} \cdot 4 \cdot 10^{-14} = 1.2 \cdot 10^{-12}.$$

Therefore, $A_3(t) = [a_1^{(3)}(t) a_2^{(3)}(t) a_3^{(3)}(t) a_4^{(3)}(t)]^T$ can be considered the exact value of the solution for the system (3.6). The analysis of the structure oscillations was performed for the value of the nonlocal parameter $\mu = 1$. In Figure 1, the configurations of the nanobeam's deflections $w2, w3$ for the iteration $m = 2$ and respectively, $m = 3$, are presented.

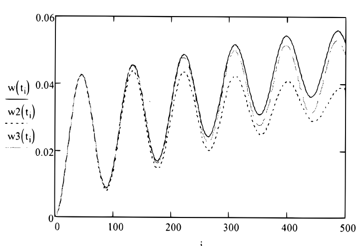

Figure 1. Comparative study between the nonlocal deflections of the nanostructure conveying fluid, w, [14] and the deflections $w2, w3$ obtained with the current algorithm for the iteration 2 and 3, $t_i = i \cdot 1.5 \cdot 10^{(-13)} s$, $i = 0, 1, \ldots, 500$.

$$w2(t_i) = \sum_{k=1}^{4} a_k^{(2)}(t_i) \varphi_k(l/2); \quad w3(t_i) = \sum_{k=1}^{4} a^{(3)})t_i) \varphi_k(l/2);$$

$t_i \cdot 1,5 \cdot 10^{-13}s$, $i = 0, 1, \ldots, 500$.

These are compared with the corresponding oscillations w obtained with the method proposed in [14]. There is a rapid convergence of the sequences corresponding to the deflections of the structure $w2, w3, \ldots$ to the solution w presented in [14].

8 Conclusions

In this paper, the forced oscillations of a nanotube conveying fluid, simply supported are studied, using an improved Zener model. The form of the constitutive law allows the approach of the presented algorithm and in the case of the dynamic analysis of microtubes. The presence of the fluid in the structure led to an inhomogeneous integral-differential system and not to an integral-differential equation for each temporal function, $a_i(t)$, as happens in the absence of fluid in the structure. The convergence of the proposed iterative method is studied theoretically and graphically. Then it is validated by a theorem. The proposed algorithm is easily accessible in the engineering design and avoids the presence of the double convolution products, which appear in the recent papers on the nanostructure conveying fluid.

Acknowledgements

The author would like to thank the editors and the referees for their valuable comments and suggestions, which greatly improved the readability and quality of the paper.

References

[1] R. Ansari, J. Torabi and M.F. Shojaei. An efficient numerical method for analyzing the thermal effects on the vibration of embedded single-walled carbon nanotubes based on the nonlocal shell model. *Mechanics of Advanced Materials and Structures*, **25**(6):500–511, 2018. https://doi.org/10.1080/15376494.2017.1285457.

[2] B. Arash and Q. Wang. A review on the application of nonlocal elastic models in modeling of carbon nanotubes and graphenes. *Computational Materials Science*, **51**(1):303–313, 2012. https://doi.org/10.1016/j.commatsci.2011.07.040.

[3] A.R. Askarian, M.R. Permoon and M. Shakouri. Vibration analysis of pipes conveying fluid resting on a fractional Kelvin-Voigt viscoelastic foundation with general boundary conditions. *International Journal of Mechanical Sciences*, **179**:105702, 2020. https://doi.org/10.1016/j.ijmecsci.2020.105702.

[4] A.R. Askarian, M.R. Permoon, M. Zahedi and M. Shakouri. Stability analysis of viscoelastic pipes conveying fluid with different boundary conditions described by fractional Zener model. *Applied Mathematical Modelling*, **103**:750–763, 2022. https://doi.org/10.1016/j.apm.2021.11.013.

[5] L. Behera and S. Chakraverty. Recent researches on nonlocal elasticity theory in the vibration of carbon nanotubes using beam models. *Archives of Computational Methods in Engineering*, **24**(3):481–494, 2017. https://doi.org/10.1007/s11831-016-9179-y.

[6] M. Cajić, D. Karličić and M. Lazarević. Nonlocal vibration of a fractional order viscoelastic nanobeam with attached nanoparticle. *Theoretical and Applied Mechanics*, **42**(3):167–190, 2015. https://doi.org/10.2298/TAM1503167C.

[7] A.C. Eringen. *Nonlocal Continuum Field Theories.* Springer-Verlag, New York, USA, 2002.

[8] S.A. Khuri and A. Sayfy. A Laplace variational iteration strategy for the solution of differential equations. *Applied Mathematics Letters*, **25**(12):2288–2305, 2012. https://doi.org/10.1016/j.aml.2012.06.020.

[9] Y. Lei, S. Adhikari and M.I. Friswell. Vibration of nonlocal Kelvin-Voight viscoelastic damped Timoshenko beams. *International Journal of Engineering Science*, **66-67**:1–13, 2013. https://doi.org/10.1016/j.ijengsci.2013.02.004.

[10] F. Mainardi. *Fractional calculus and waves in linear viscoelasticity.* Imperial College Press, London, 2010. https://doi.org/10.1142/p614.

[11] F. Mainardi and G. Spada. Creep, relaxation and viscosity properties for basic fractional models in rheology. *The European Physical Journal Special Topics*, **193**:133–160, 2011. https://doi.org/d10.1140/epjst/e2011-01387-1.

[12] I. Maron and B. Demidovich. *Numerical Calculation Elements.* Edition MIR, Moscow, 1973.

[13] O. Martin. Nonlinear dynamic analysis of viscoelastic beams using a fractional rheological model. *Applied Mathematical Modelling*, **43**:351–359, 2017. https://doi.org/10.1016/j.apm.2016.11.033.

[14] O. Martin. An iterative algorithm for studying forced vibrations of a nanotube conveying fluid. *Mechanics of Advanced Materials and Structures*, **29**(25):4180–4192, 2022. https://doi.org/10.1080/15376494.2021. 1921317.

[15] J. Peddieson, G.R. Buchanan and R.P. McNitt. Application of nonlocal continuum models to nanotechnology. *International Journal of Engineering Science*, **41**(3):305–312, 2003. https://doi.org/10.1016/S0020-7225(02)00210-0.

[16] J.N. Reddy and S.D. Pang. Nonlocal continuum theories of beams for the analysis of carbon nanotubes. *Journal of Applied Physics*, **103**(2):23511–23527, 2008. https://doi.org/10.1063/1.2833431.

[17] S. Shaw, M.K. Warby and J.R. Whiteman. A comparison of hereditary integral and internal variable approaches to numerical linear solid viscoelasticity. In *Proceedings of the Fourteenth Polish Conference on Computer Methods in Mechanics, Poznan, 1997*, pp. 183–200, 1997. https://doi.org/dml.cz/dmlcz/700266.

[18] Y.Q. Wang, H.H. Li, Y.F. Zhang and J.W. Zu. A nonlinear surface-stress-dependent model for vibration analysis of cylindrical nanoscale shells conveying fluid. *Applied Mathematical Modelling*, **64**(12):55–70, 2018. https://doi.org/10.1016/j.apm.2018.07.016.

[19] Y.Q. Wang, Y.H. Wan and J.W. Zu. Nonlinear dynamic characteristics of functionally graded sandwich thin nanoshells conveying fluid incorporating surface stress influence. *Thin-Walled Structures*, **135**(2):537–547, 2019. https://doi.org/10.1016/j.tws.2018.11.023.