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# A Fixed-Point Type Result for Some Non-Differentiable Fredholm Integral Equations

**Miguel A. Hernández-Verón[a], Sukhjit Singh[b], Eulalia Martínez[c] and Nisha Yadav[b]**

[a] *Department of Mathematics and Computation, University of La Rioja*
Logroño, Spain
[b] *Department of Mathematics, Dr. B.R. Ambedkar National Institute of Technology*
Jalandhar, India
[c] *Instituto Universitario de Matemática Multidisciplinar, Universitat Politècnica de València*
València, Spain

E-mail(*corresp.*): eumarti@mat.upv.es
E-mail: mahernan@unirioja.es
E-mail: kundalss@nitj.ac.in
E-mail: nisha.ma.21@nitj.ac.in

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**Abstract.** In this paper, we present a new fixed-point result to draw conclusions about the existence and uniqueness of the solution for a nonlinear Fredholm integral equation of the second kind with non-differentiable Nemytskii operator. To do this, we will transform the problem of locating a fixed point for an integral operator into the problem of locating a solution of an integral equation. Thus, assuming conditions on the Nemytskii operator, we will obtain a global convergence domain for the solution of the considered integral equation, taking for this a uniparametric family of derivative-free iterative processes with quadratic convergence. This result provides us a new fixed-point result for the integral operator considered.

**Keywords:** fixed point theorem, global convergence, Fredholm integral equations, derivative-free iterative processes.

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## 1 Introduction

Nowadays, numerical analysis is a very important branch in mathematics due to the fact that a great variety of applied problems in science, engineering, computer science, bio-medicine etc., can be formulated by using ordinary differential equations, partial differential equations, nonlinear integrals or just nonlinear equations. It is well known that these kind of equations rarely have algebraic solution, so we have to solve these problems by numerical methods. Moreover, the generalized use of the computer has improved the behavior of the approximated solutions obtained in these cases, because of the improvements in stability, precision and computational time used.

We focus now in integral equations of Fredholm-type, different numerical methods have been developed to approximate their solution, for example, Fredholm-type integral equations [18, 21, 24], Volterra-Fredholm integral equations [9, 17] and nonlinear Fredholm integro-differential equations [16] can be considered.

Let us consider a special type of the nonlinear Fredholm integral equation of the second kind [7, 8]

$$x(t) = g(t) + \beta \int_b^c G(t,s)\mathcal{P}(x)(s)ds, \quad t \in [b,c], \tag{1.1}$$

where $\beta \in \mathbb{R}$, $-\infty < b < c < +\infty$, $g : [b,c] \to \mathbb{R}$, $G : [b,c] \times [b,c] \to \mathbb{R}$ are continuous functions and $x : [b,c] \to \mathbb{R}$ is the unknown function to be determined in $\mathcal{C}([b,c])$. The Nemytskii operator $\mathcal{P} : \Omega \subseteq \mathcal{C}([b,c]) \to \mathcal{C}([b,c])$ is given by $\mathcal{P}(x)(s) = P(x(s))$, where $\Omega$ is a nonempty open convex domain in $\mathcal{C}([b,c])$ and $P : \mathbb{R} \to \mathbb{R}$ is a continuous but non-differentiable function. The set $\mathcal{C}([b,c])$ denotes the space of continuous real functions in $[b,c]$, which is a Banach space with the infinity norm that we will use.

In order to approximate their solution we can use different numerical methods. To begin with the homotopy analysis method (see [4, 9]). Secondly, we want to mention the hybrid method proposed by the Adomian decomposition, [6]. The technique based on a discretization process for the integral equation is also interesting (see [3, 19, 22]). Finally, we center in the use of iterative schemes to approximate the solutions. The best known methods in this case are Newton-type methods, Whittaker-type methods or higher order convergence iterative schemes, see [13, 22]. The use of this technique has successfully permitted the obtainment of existence and uniqueness domains for the solution of nonlinear integral equations by setting an adequate theoretical semilocal convergence study.

If we consider the integral operator $\mathcal{H} : \Omega \subset \mathcal{C}([b,c]) \to \mathcal{C}([b,c])$, given by

$$x(t) = [\mathcal{H}(x)](t) \quad \text{with} \quad [\mathcal{H}(x)](t) = g(t) + \beta \int_b^c G(t,s)\mathcal{P}(x)(s)ds, \tag{1.2}$$

then, the Banach fixed point theorem guarantees that, under certain assumptions, $\mathcal{H}$ has a unique fixed point, thus, the Fredholm integral equation (1.1) has exactly one solution. Besides, the method of successive approximations:

$$\begin{cases} x_0 \text{ given in } \mathcal{C}([b,c]), \\ x_{n+1} = \mathcal{H}(x_n), \quad n \geq 0, \end{cases} \tag{1.3}$$

converges globally to the solution $x^*$ of (1.1). But this scheme has some difficulties: the rate of convergence of sequence of successive approximations $\{x_n\}$ is slow, $\mathcal{H}$ must be contractive and, crucially, the integral equation (1.1) must have a unique fixed point at $\mathcal{C}([b,c])$, which in the nonlinear case at hand is not usually a feature of the equation. On the contrary, a very favorable feature is that the method of successive approximations is globally convergent.

If we want to consider other possible situations in which the operator $\mathcal{H}$ has more fixed points, we must apply a restricted fixed point theorem, as for example:

**Theorem 1.** *( [5]) If $\mathcal{D}$ is a convex and compact set of $\mathcal{C}([b,c])$ and the operator $\mathcal{H} : \mathcal{D} \to \mathcal{D}$ is a contraction, then $\mathcal{H}$ has a unique fixed point $x^*$ in $\mathcal{D}$ that can be approximated by the method of successive approximations* (1.3) *from any starting point $x_0$ given in $\mathcal{D}$.*

In this case, the first problem is obviously to locate a domain $\mathcal{D}$ that contains a fixed point of the operator $\mathcal{H}$. For this, we need some information about the possible fixed points of the operator $\mathcal{H}$.

To remove these difficulties, we observe that Equation (1.1) can be written as $\mathcal{K}(x) = 0$, for $\mathcal{K} : \Omega \subseteq \mathcal{C}([b,c]) \to \mathcal{C}([b,c])$, where

$$[\mathcal{K}(x)](t) = x(t) - g(t) - \beta \int_b^c G(t,s)\mathcal{P}(x)(s)ds, \quad t \in [b,c]. \tag{1.4}$$

Obviously, a solution of equation $\mathcal{K}(x) = 0$ is a fixed point of operator $\mathcal{H}$ and, therefore, a solution of nonlinear integral equation (1.1). Taking into account this operator, we will be able to transfer the problem of obtaining a fixed point of the operator $\mathcal{H}$ to the problem of obtaining a solution of the equation $\mathcal{K}(x) = 0$.

The main objective of our work is to obtain a restricted fixed point result for the operator $\mathcal{H}$, like the one cited above, but in such a way that the approximation of the fixed point is not carried out using the method of successive approximations, of linear convergence, but rather let us do it through an iterative process with quadratic convergence. Furthermore, we are interested in obtaining some kind of condition that allows us to locate the domain $\mathcal{D}$. To achieve this goal, our strategy will consist of considering an iterative process with quadratic convergence and obtaining a global convergence result for said iterative process. As is known, the study of the global convergence of an iterative process is a difficult problem to deal with. In our case, we will use the technique in which an auxiliary point is considered to ensure the global convergence.

The organization of the paper is as follows. Section 2 presents the iterative schemes used to approximate a solution of the integral equation. We also define a first-order divided difference for the Nemytskii operator, which will allow us to apply the considered iterative schemes later. Next, in Section 3, a global convergence study is performed by giving first some auxiliary lemmas before setting the main theorems and the corresponding improvements. Finally, Section 4 shows a numerical experiment where we illustrate the obtained results. We conclude giving some final remarks.

We denote $\mathcal{B}(\tilde{u},R) = \{u \in \mathcal{C}([b,c]); \|u-\tilde{u}\| < R\}$ and $\overline{\mathcal{B}(\tilde{u},R)} = \{u \in \mathcal{C}([b,c]); \|u-\tilde{u}\| \leq R\}$ for open and closed balls with center $\tilde{u}$ and radius $R>0$.

## 2 Preliminaries

To obtain a global convergence result for an iterative process, a technique used is to apply a fixed point theorem, with the limitations that this result poses. However, we will follow the ideas of the result given in [8], in which the authors established a global convergence domain for Newton's method, with quadratic convergence, when it is applied to nonlinear differentiable integral equations of type (1.1). In our case, Equation (1.1) is nondifferentiable, so we have to consider a derivative-free iterative process. To obtain derivative-free iterative processes, it is general to approximate the derivatives by divided differences [1,10]. If we denote the space of bounded linear operators from $\Omega$ to $\mathcal{C}([b,c])$ by $L(\Omega,\mathcal{C}([b,c]))$, then an operator $[u,v;\mathcal{D}] \in L(\Omega,\mathcal{C}([b,c]))$ is called a first-order divided difference for the operator $\mathcal{F}: \Omega \subseteq \mathcal{C}([b,c]) \to \mathcal{C}([b,c])$ on the points $u$ and $v$ $(u \neq v)$ if

$$[u,v;\mathcal{F}](u-v) = \mathcal{F}(u) - \mathcal{F}(v).$$

Thus, if we consider Newton's method for the operator $\mathcal{K}$:

$$\begin{cases} x_0 \text{ given in } \mathcal{C}([b,c]), \\ x_{n+1} = x_n - [\mathcal{K}'(x_n)]^{-1}\mathcal{K}(x_n), \quad n \geq 0, \end{cases}$$

in [14], the authors consider the following approximation

$$\mathcal{K}'(x_n) \sim [(1-\mu)x_n + \mu x_{n-1}, (1+\mu)x_n - \mu x_{n-1}; \mathcal{K}], \text{ for } \mu > 0,$$

and they consider the following derivative-free uniparametric family of iterative processes,

$$\begin{cases} x_0, x_{-1} \text{ given in } \Omega, \ \mu \in [0,1], \\ y_n = (1-\mu)x_n + \mu x_{n-1}, \\ z_n = (1+\mu)x_n - \mu x_{n-1}, \\ x_{n+1} = x_n - [y_n, z_n; \mathcal{K}]^{-1} \mathcal{K}(x_n), \quad n \geq 0. \end{cases} \tag{2.1}$$

Notice that the family (2.1) can be assumed as a combination of the Newton's method ($\mu = 0$) for differentiable case and the Kurchatov method ($\mu = 1$) in both cases, differentiable and non-differentiable for operator $\mathcal{K}$. This uniparametric family maintains the quadratic convergence [14] as the Kurchatov method [2,23] and improves the accessibility of the Kurchatov method by considering values near to $\mu = 0$, which is similar to the Newton's method.

Hernández *et al.* [14] established the local and semilocal convergence analysis of method (2.1) for non-differentiable operators under $\omega$-conditions.

Our main goal of this work is to study the integral equation of type (1.1) from the iterative process (2.1). We obtain the domain of global convergence for method (2.1) for a fixed $\mu \in (0,1]$. For this, we assume conditions on the Nemytskii operator $\mathcal{P}$ and consider an auxiliary point in $\Omega$. From this auxiliary point, we prove the existence and uniqueness of a solution $x^*$ of the integral equation $\mathcal{K}(x) = 0$ and ensure the convergence of (2.1), starting from any point in the ball centered on an auxiliary point that we consider. This will allow us to obtain the restricted global convergence result for the operator $\mathcal{H}$.

Now, keeping in mind the iterative processes given in (2.1), we need to define a first-order divided difference $[u, v; \mathcal{K}]$ for the application of iterative schemes given by (2.1). So, for given continuous real functions $u$ and $v$ ($u \neq v$), we define $[u, v; \mathcal{K}] : \Omega \subseteq \mathcal{C}([b, c]) \to \mathcal{C}([b, c])$ with

$$[u, v; \mathcal{K}](w)(t) = w(t) - \beta \int_b^c G(t, s)[u, v; \mathcal{P}](w)(s)ds,$$

where

$$[u, v; \mathcal{P}](w)(s) = \begin{cases} \dfrac{P(u(s)) - P(v(s))}{u(s) - v(s)} w(s) & \text{if } s \in [b, c] \text{ with } u(s) \neq v(s), \\ 0 & \text{if } s \in [b, c] \text{ with } u(s) = v(s), \end{cases}$$

is obviously a first-order divided difference in $\Omega$. Then, $[u, v; \mathcal{K}] \in L(\Omega, \mathcal{C}([b, c]))$ and for $u \neq v$, with $u, v \in \mathcal{C}([b, c]))$ it is easy to check that

$$[u, v; \mathcal{K}](u - v) = \mathcal{K}(u) - \mathcal{K}(v),$$

and, therefore the operator $[u, v; \mathcal{P}]$ allows us to define a first-order divided difference $[u, v; \mathcal{K}]$ for the operator $\mathcal{K} : \Omega \subseteq \mathcal{C}([b, c]) \to \mathcal{C}([b, c])$.

## 3 Main results

This section examines the global convergence of iterative scheme (2.1) for the operator equation $\mathcal{K}(x) = 0$, with $\mathcal{K}$ given in (1.4). A qualitative property of the study of the convergence of iterative processes is obtaining a result of the existence of a solution. This fact, combined with the ad hoc elaboration of uniqueness result, will allow us to obtain a result of existence and uniqueness of a fixed point for the operator $\mathcal{H}$.

### 3.1 Global convergence and uniqueness of solution for the iterative processes given in (2.1)

The analysis of the global convergence is based on demanding conditions only on the operator $\mathcal{K}$ that ensure the convergence to a solution $x^*$ of the equation $\mathcal{K}(x) = 0$. For a fixed value of $\mu \in (0, 1]$, we will establish the global convergence result for the iterative process (2.1) under the following condition for the first-order divided difference of the Nemystkii operator

**(I)** $\|[u,v;\mathcal{P}]-[x,y;\mathcal{P}]\| \leq A+L(\|u-x\|+\|v-y\|)$, for pairs of distinct points $(u,v),(x,y)\in \Omega\times\Omega \subseteq \mathcal{C}([b,c])\times\mathcal{C}([b,c])$, with $A\geqslant 0$ and $L\geqslant 0$.

Notice that, with respect to the first-order divided difference of the Nemystkii operator, we include the boundedness that is used in the non-diferenctiable case $A>0$ (see [12]), and if $A=0$ this condition **(I)** is a generalization of the case in which $[x,y;\mathcal{P}]$ is Lipschitz-continuous condition ( [15]). Notice that, the above case, the Fréchet derivative of $\mathcal{P}$ exists in $\Omega$, see [12], and satisfies $[x,x;\mathcal{P}]=\mathcal{P}'(x)$, see [1]. So, if $A=0$ then $\mathcal{P}$ is differentiable. Therefore, the results that we will obtain will be valid both for the case of the non-differentiable Nemystkii operator $(A>0)$ and for the differentiable Nemystkii operator case $(A=0)$.

Obviously, from condition **(I)**, it follows that fixed a pair of distinct points $(\tilde{u},\tilde{v})\in\Omega\times\Omega$, there exist $\widetilde{A}\geq 0$ and $\widetilde{L}\geq 0$, such that for each pair of distinct points $(u,v)\in\Omega\times\Omega$,

$$\|[u,v;\mathcal{P}]-[\tilde{u},\tilde{v};\mathcal{P}]\| \leq \widetilde{A}+\widetilde{L}(\|u-\tilde{u}\|+\|v-\tilde{v}\|),$$

with $\widetilde{A}\leq A$ and $\widetilde{L}\leq L$.

Moreover, by using previous conditions, we can achieve the following result:

**Lemma 1.** *Under condition **(I)**, the following results are verified:*

*(a) For pairs of distinct points $(u,v),(x,y)\in\Omega\times\Omega$,*

$$\|[u,v;\mathcal{K}]-[x,y;\mathcal{K}]\| \leq \beta M(A+L(\|u-x\|+\|v-y\|)),$$

*with $M=\|\int_b^c G(t,s)ds\|$.*

*(b) Fixed a pair of distinct points $(\tilde{u},\tilde{v})\in\Omega\times\Omega$, there exist $\widetilde{A}\geq 0$ and $\widetilde{L}\geq 0$, such that for each pair of distinct points $(u,v)\in\Omega\times\Omega$,*

$$\|[u,v;\mathcal{K}]-[\tilde{u},\tilde{v};\mathcal{K}]\| \leq \beta M(\widetilde{A}+\widetilde{L}(\|u-\tilde{u}\|+\|v-\tilde{v}\|)),$$

*with $\widetilde{A}\leq A$ and $\widetilde{L}\leq L$.*

Next, we have two aspects to complete in our hypotheses. On the one hand, the hypothesis related to the auxiliary point $\tilde{u}\in\Omega$ that we will consider. Besides, taking into account that the iterative processes given in (2.1) are with memory, we have to place the initial value $x_{-1}$. Therefore, we take $\tilde{v}=x_{-1}\in\Omega$ with $x_{-1}\neq\tilde{u}$ and $\|x_{-1}-\tilde{u}\|\leq\alpha$ for $\alpha>0$. On the other hand, we have to indicate an expression that allows us to calculate the radius $R$ of the ball of existence and uniqueness of solution, which will also provide us with the global convergence of the iterative processes (2.1). Moreover, we will also need to ensure that the sequences of iterations $\{x_n\},\{y_n\}$ and $\{z_n\}$ given in (2.1), are well defined in $\Omega$ and $\{x_n\}$ converges to a solution of the equation $\mathcal{K}(x)=0$.

Thus, we will also consider the following hypotheses:

**(II)** There exists $\widetilde{\Gamma}=[\tilde{u},x_{-1};\mathcal{K}]^{-1}$ such that $\|\widetilde{\Gamma}\|\leq\lambda$ and $\|[\tilde{u},x_{-1};\mathcal{K}]^{-1}\mathcal{K}(\tilde{u})\|\leq\theta$, with $\theta>0$.

**(III)** The auxiliary real equation

$$\theta = (1 - h(t) - \widetilde{h}(t))t, \tag{3.1}$$

where $h(t) = \lambda\beta M(A+L(1+4\mu)t)$ and $\widetilde{h}(t) = \lambda\beta M(\widetilde{A}+\widetilde{L}(2t(1+\mu)+\alpha))$ has at least one positive real root and we denote by $R$ the smallest positive real root.

**(IV)** $\overline{B(\tilde{u},(1+2\mu)R)} \subset \Omega$ and $\lambda\beta M(A+2LR(1+\mu)) + \widetilde{h}(R) < 1$.

Considering the preceding notation, we shall show the main result of global convergence for iterative processes (2.1) based on conditions **(I)** – **(IV)**. Previously, notice that we will consider that $x_{k+1} \neq x_k$ for all $k \geqslant 0$, because, in other case, $x_{k+1} = x_k$ for some $k \geqslant 0$, and then the sequence $\{x_n\}$ converges to $x^*$ with $x^* = x_n = x_{k+1} = x_k$ for all $n \geqslant k+2$. Moreover, if $x_{k+1} \neq x_k$ we obtain that $y_{k+1} \neq z_{k+1}$. Therefore, the operators $[y_{k+1}, z_{k+1}; \mathcal{K}]$ are always well defined, for $k \geqslant 0$, if $y_{k+1}, z_{k+1} \in \Omega$.

We will begin our convergence study by considering $n = 0$. Let $x_0 \in \overline{B(\tilde{u},R)}$, with $x_0 \neq x_{-1}$ and $x_0 \neq \tilde{u}$, then by definition of method (2.1) and hypotheses **(II)** and **(III)**, we obtain

$$\begin{aligned}
\|y_0 - \tilde{u}\| &\leq (1-\mu)\|x_0 - \tilde{u}\| + \mu\|x_{-1} - \tilde{u}\| \\
&\leq (1-\mu)R + \mu\alpha \leq (1-\mu)R + \mu R = R, \\
\|z_0 - \tilde{u}\| &\leq (1+\mu)\|x_0 - \tilde{u}\| + \mu\|x_{-1} - \tilde{u}\| \leq (1+\mu)R + \mu\alpha \leq (1+2\mu)R,
\end{aligned} \tag{3.2}$$

it follows that $y_0 \in B(\tilde{u},R) \subset \Omega$ and $z_0 \in B(\tilde{u},(1+2\mu)R) \subset \Omega$ and, as $y_0 \neq z_0$, $[y_0, z_0; \mathcal{K}]$ is well defined. Then, from hypotheses **(III)** and **(IV)**

$$\begin{aligned}
&\|I - [\tilde{u}, x_{-1}; \mathcal{K}]^{-1}[y_0, z_0; \mathcal{K}]\| \leq \|[\tilde{u}, x_{-1}; \mathcal{K}]^{-1}\|\|[\tilde{u}, x_{-1}; \mathcal{K}] - [y_0, z_0; \mathcal{K}]\| \\
&\leq \lambda\beta M(\widetilde{A} + \widetilde{L}(\|y_0 - \tilde{u}\| + \|z_0 - x_{-1}\|)) \leq \lambda\beta M(\widetilde{A} + \widetilde{L}(R + (1+2\mu)R + \alpha)) \\
&\leq \lambda\beta M(\widetilde{A} + \widetilde{L}(2R(1+\mu) + \alpha)) = \widetilde{h}(R) < 1,
\end{aligned}$$

since from (3.1), as $R > 0$, we get that $h(R) + \widetilde{h}(R) < 1$, and then $\widetilde{h}(R) < 1$. Therefore, by applying Banach Lemma, $[y_0, z_0; \mathcal{K}]^{-1}$ exists with

$$\|[y_0, z_0; \mathcal{K}]^{-1}[\tilde{u}, x_{-1}; \mathcal{K}]\| \leq \frac{1}{1 - \widetilde{h}(R)}, \quad \|[y_0, z_0; \mathcal{K}]^{-1}\| \leq \frac{\lambda}{1 - \widetilde{h}(R)}.$$

Next, as

$$\mathcal{K}(x_0) = \mathcal{K}(\tilde{u}) + [\tilde{u}, x_{-1}; \mathcal{K}](x_0 - \tilde{u}) + ([x_0, \tilde{u}; \mathcal{K}] - [\tilde{u}, x_{-1}; \mathcal{K}])(x_0 - \tilde{u}),$$

from Lemma 1, we get

$$\begin{aligned}
\|[\tilde{u}, x_{-1}; \mathcal{K}]^{-1}\mathcal{K}(x_0)\| &\leq \|[\tilde{u}, x_{-1}; \mathcal{K}]^{-1}\mathcal{K}(\tilde{u})\| + \|x_0 - \tilde{u}\| \\
&\quad + \|[\tilde{u}, x_{-1}; \mathcal{K}]^{-1}\|\|[x_0, \tilde{u}; \mathcal{K}] - [\tilde{u}, x_{-1}; \mathcal{K}]\|\|x_0 - \tilde{u}\| \\
&\leq \theta + R + \lambda\beta M(\widetilde{A} + \widetilde{L}(R + \alpha))R.
\end{aligned}$$

Therefore,

$$\|x_1 - x_0\| \le \|[y_0, z_0; \mathcal{K}]^{-1}\mathcal{K}(x_0)\| \le \|[y_0, z_0; \mathcal{K}]^{-1}[\tilde{u}, x_{-1}; \mathcal{K}]\| \\ \times \|[\tilde{u}, x_{-1}; \mathcal{K}]^{-1}\mathcal{K}(x_0)\| \le \frac{\theta + R + \lambda\beta M(\widetilde{A} + \widetilde{L}(R+\alpha))R}{1 - \widetilde{h}(R)}, \tag{3.3}$$

and as $\|x_0 - y_0\| \le \mu\|x_0 - x_{-1}\| \le \mu R + \mu\alpha$, from (3.2), (3.3) and hypothesis **(II)**, we obtain

$$\begin{aligned}
\|x_1 - \tilde{u}\| &\le \|[y_0, z_0; \mathcal{K}]^{-1}\big(([x_0, \tilde{u}, \mathcal{K}] - [y_0, z_0; \mathcal{K}])(x_0 - \tilde{u}) + \mathcal{K}(\tilde{u})\big)\| \\
&\le \|[y_0, z_0; \mathcal{K}]^{-1}\|\|[x_0, \tilde{u}; \mathcal{K}] - [y_0, z_0; \mathcal{K}]\|\|x_0 - \tilde{u}\| \\
&\quad + \|[y_0, z_0; \mathcal{K}]^{-1}[\tilde{u}, x_{-1}; \mathcal{K}]\|\|[\tilde{u}, x_{-1}; \mathcal{K}]^{-1}\mathcal{K}(\tilde{u})\| \\
&\le \frac{\theta + \lambda\beta M(A + L(\|x_0 - y_0\| + \|\tilde{u} - z_0\|))R}{1 - \widetilde{h}(R)} \\
&\le \frac{\theta + \lambda\beta M(A + L(\mu R + \mu\alpha + (1+2\mu)R))R}{1 - \widetilde{h}(R)} \\
&\le \frac{\theta + \lambda\beta M(A + L((1+3\mu)R + \mu\alpha))R}{1 - \widetilde{h}(R)} \le \frac{\theta + h(R)R}{1 - \widetilde{h}(R)} = R.
\end{aligned}$$

Thus, $x_1 \in \overline{B(\tilde{u}, R)}$.

After that, we also give some certain properties that will be used later.

**Lemma 2.** *For the sequence* $\{x_n\}$ *defined by* (2.1)*, we have*

$$\begin{aligned}
&\mathcal{K}(x_n) = ([x_n, x_{n-1}; \mathcal{K}] - [y_{n-1}, z_{n-1}; \mathcal{K}])(x_n - x_{n-1}), \\
&\|x_n - \tilde{u}\| \le \|[y_{n-1}, z_{n-1}; \mathcal{K}]^{-1}([y_{n-1}, z_{n-1}; \mathcal{K}] - [x_{n-1}, \tilde{u}; \mathcal{K}])(x_{n-1} - \tilde{u})\| \\
&\qquad + \|[y_{n-1}, z_{n-1}; \mathcal{K}]^{-1}\mathcal{K}(\tilde{u})\|,
\end{aligned}$$

*for all* $x_n, x_{n-1} \in \overline{B(\tilde{u}, R)}$*, with* $x_n \ne x_{n-1}$*,* $y_{n-1} \in \overline{B(\tilde{u}, R)}$*,* $z_{n-1} \in \overline{B(\tilde{u}, (1+2\mu)R)}$ *and* $y_{n-1} \ne z_{n-1}$*.*

**Lemma 3.** *Assume that the conditions **(I)** – **(IV)** are satisfied. Then,* $[u, v; \mathcal{K}]^{-1}$ *exists with*

$$\|[u, v, \mathcal{K}]^{-1}[\tilde{u}, x_{-1}; \mathcal{K}]\| \le \frac{1}{1 - \widetilde{h}(R)}, \quad \|[u, v; \mathcal{K}]^{-1}\| \le \frac{\lambda}{1 - \widetilde{h}(R)},$$

*for pair of distinct points* $(u, v) \in \overline{B(\tilde{u}, R)} \times \overline{B(\tilde{u}, (1+2\mu)R)}$*.*

*Proof.* From Lemma 1 and condition **(IV)**, we obtain

$$\begin{aligned}
&\|I - [\tilde{u}, x_{-1}; \mathcal{K}]^{-1}[u, v; \mathcal{K}]\| \le \|[\tilde{u}, x_{-1}; \mathcal{K}]^{-1}\|\|[\tilde{u}, x_{-1}; \mathcal{K}] - [u, v; \mathcal{K}]\| \\
&\le \lambda\beta M(\widetilde{A} + \widetilde{L}(\|u - \tilde{u}\| + \|v - x_{-1}\|)) \le \lambda\beta M(\widetilde{A} + \widetilde{L}(R + (1+2\mu)R + \alpha)) \\
&\le \lambda\beta B(\widetilde{A} + \widetilde{L}(2R(1+2\mu) + \alpha)) = \widetilde{h}(R) < 1,
\end{aligned}$$

therefore, by applying Banach Lemma, $[u, v; \mathcal{K}]^{-1}$ exists and the results are satisfied.

Next, we consider $n = 1$. Then, from (2.1), we obtain

$$\|y_1 - \tilde{u}\| \leq (1-\mu)\|x_1 - \tilde{u}\| + \mu\|x_0 - \tilde{u}\| < R,$$
$$\|z_1 - \tilde{u}\| = \|x_1 - \tilde{u}\| + \mu\|x_1 - x_0\| < R + 2\mu R = (1+2\mu)R,$$

then, it follows that $y_1 \in \overline{B(\tilde{u}, R)} \subset \Omega$, and $z_1 \in \overline{B(\tilde{u}, (1+2\mu)R)} \subset \Omega$. As $y_1 \neq z_1$ then $[y_1, z_1; \mathcal{K}]$ is well defined. In addition, as $\|x_0 - z_0\| \leq \mu\|x_0 - x_{-1}\| \leq \mu(R + \alpha) \leq 2\mu R$ and from Lemma 2, we get

$$\begin{aligned}\|\mathcal{K}(x_1)\| &\leq \beta M(A + L(\|x_1 - y_0\| + \|x_0 - z_0\|))\|x_1 - x_0\| \\ &\leq \beta M(A + L(2R + 2\mu R))\|x_1 - x_0\| \\ &\leq \beta M(A + L(2(1+\mu)R)\|x_1 - x_0\|.\end{aligned}$$

Now, by using (2.1), we obtain

$$\begin{aligned}&\|x_2 - x_1\| \leq \|[y_1, z_1; \mathcal{K}]^{-1}\mathcal{K}(x_1)\| \\ \leq &\frac{\lambda\beta M(A + L(2(1+\mu)R))}{1 - \widetilde{h}(R)}\|x_1 - x_0\| \leq N\|x_1 - x_0\|,\end{aligned}$$

where, $N = \frac{\lambda\beta M(A+L(2(1+\mu)R))}{1-\widetilde{h}(R)} < 1$ from **(IV)**. Furthermore, as $\|x_1 - y_1\| \leq \mu\|x_1 - x_0\| \leq 2\mu R$, we have

$$\begin{aligned}\|x_2 - \tilde{u}\| &\leq \|[y_1, z_1; \mathcal{K}]^{-1}([x_1, \tilde{u}; \mathcal{K}] - [y_1, z_1; \mathcal{K}])(x_1 - \tilde{u})\| + \|[y_1, z_1; \mathcal{K}]^{-1}\mathcal{K}(\tilde{u})\| \\ &\leq \frac{\theta + \lambda\beta M(A + L(2\mu R + (1+2\mu)R))R}{1 - \widetilde{h}(R)} = \frac{\theta + h(R)R}{1 - \widetilde{h}(R)} = R,\end{aligned}$$

so, $x_2 \in \overline{B(\tilde{u}, R)} \subseteq \Omega$. □

Next, to generalize the results obtained in steps $n = 0$ and $n = 1$, we establish the recurrence relations that verify the elements of the sequence $\{x_n\}$ given in (2.1).

**Lemma 4.** *Under conditions **(I)**–**(IV)**, the following recurrence relations are verified for $n \geq 2$.*

**(i)** $\|\mathcal{K}(x_n)\| \leq \beta M(A + 2LR(1+\mu))\|x_n - x_{n-1}\|$.

**(ii)** $\|x_{n+1} - x_n\| \leq N\|x_n - x_{n-1}\| \leq N^2\|x_{n-1} - x_{n-2}\| \ldots \leq N^n\|x_1 - x_0\| < \|x_1 - x_0\|$.

**(iii)** $\|x_{n+1} - \tilde{u}\| \leq (\theta + h(R)R)/\big(1 - \widetilde{h}(R)\big) = R$.

*Proof.* We can prove these recurrence relation by using mathematical induction. Thus, for $n = 2$ we have

$$\|y_2 - \tilde{u}\| \leq (1-\mu)\|x_2 - \tilde{u}\| + \mu\|x_1 - \tilde{u}\| \leq R,$$
$$\|z_2 - \tilde{u}\| = \|x_2 - \tilde{u}\| + \mu\|x_2 - x_1\| \leq (1+2\mu)R,$$

then, it follows that $y_2 \in \overline{B(\tilde{u}, R)} \subset \Omega$, $z_2 \in \overline{B(\tilde{u}, (1+2\mu)R)} \subset \Omega$ and, as $y_2 \neq z_2$, then $[y_2, z_2; \mathcal{K}]$ is well defined. Then, from Lemma 2, and taking into account that

$$\begin{aligned} &\|x_2 - y_1\| \le \|x_2 - \tilde{u}\| + \|y_1 - \tilde{u}\| \le 2R, \\ &\|x_1 - z_1\| = \|x_1 - (1+\mu)x_1 + \mu x_0\| \le \mu\|x_1 - x_0\| < 2\mu R, \end{aligned}$$

we get

$$\begin{aligned} \|\mathcal{K}(x_2)\| &\le \beta M(A + L(\|x_2 - y_1\| + \|x_1 - z_1\|))\|x_2 - x_1\| \\ &\le \beta M(A + L(2(1+\mu)R))\|x_2 - x_1\|. \end{aligned}$$

Thus, item $(i)$ is verified for $n = 2$. On the other hand, it follows that

$$\begin{aligned} \|x_3 - x_2\| \le \|[y_2, z_2; \mathcal{K}]^{-1}\mathcal{K}(x_2)\| &\le \frac{\lambda\beta M(A + L(2(1+\mu)R))}{1 - \widetilde{h}(R)} \\ \times \|x_2 - x_1\| &\le N\|x_2 - x_1\| \le N^2\|x_1 - x_0\|, \end{aligned}$$

so, item $(ii)$ is verified for $n = 2$. Next, from (3.1) and Lemma 2, as

$$\|y_2 - x_2\| = \|(1-\mu)x_2 + \mu x_1 - x_2\| \le \mu\|x_2 - x_1\| < 2\mu R,$$

we obtain

$$\begin{aligned} \|x_3 - \tilde{u}\| &\le \|[y_2, z_2; \mathcal{K}]^{-1}([y_2, z_2; \mathcal{K}] - [x_2, \tilde{u}; \mathcal{K}])(x_2 - \tilde{u})\| + \|[y_2, z_2; \mathcal{K}]^{-1}\mathcal{K}(\tilde{u})\| \\ &< \frac{\lambda\beta M(A + L(2\mu R + R(1+2\mu)))R + \theta}{1 - \widetilde{h}(R)} = \frac{\theta + h(R)R}{1 - \widetilde{h}(R)} = R, \end{aligned}$$

then, the item $(iii)$ is verified for $n = 2$.

Next, applying a process of mathematical induction, the result is proved in a simple way. Since the inductive step is proved analogous to the one used for the step $n = 2$. $\square$

Once the recurrence relations seen in the previous result have been proven, we are now in a position to prove the global convergence result for the iterative processes given in (2.1) applied to the equation $\mathcal{K}(x) = 0$.

**Theorem 2.** *Let us assume that conditions* ***(I)*** *–* ***(IV)*** *are verified. Then, the iterative process* (2.1) *is well defined and converges to* $x^*$*, a solution of* $\mathcal{K}(x) = 0$*, with* $x_n, x^* \in \overline{B(\tilde{u}, R)}$ *for any initial point* $x_0 \in \overline{B(\tilde{u}, R)}$*, with* $x_0 \neq \tilde{u}$*. Moreover,* $x^*$ *is the only solution of the equation* $\mathcal{K}(x) = 0$ *in* $\overline{B(\tilde{u}, R)}$*.*

*Proof.* From the recurrence relations, we have to prove that the sequence $\{x_n\}$ is a Cauchy sequence in the Banach space $\mathcal{C}([b, c])$, then, $\{x_n\}$ is a convergent sequence. For this, we consider

$$\begin{aligned} &\|x_{n+m} - x_n\| \le \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \dots \\ &+ \|x_{n+2} - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &< (N^{n+m-1} + N^{n+m-2} + \dots + N^{n+1} + N^n)\|x_1 - x_0\| < \frac{N^n}{1 - N}\|x_1 - x_0\|. \end{aligned}$$

Then, obviously, $\{x_n\}$ is a Cauchy sequence. Therefore, there exists $x^*$ such that $\{x_n\} \to x^*$. On the other hand, from Lemma 4, we have that $\|\mathcal{K}(x_n)\| \leq \beta M(A+2LR(1+\mu))\|x_n - x_{n-1}\|$ and, by continuity of operator $\mathcal{K}$, when $n \to \infty$ we get $\mathcal{K}(x^*) = 0$.

To finish, we will prove the uniqueness of the solution $x^*$. Let $y^*$ be another solution of equation $\mathcal{K}(x) = 0$ in $\overline{B(\tilde{u}, R)}$. We consider

$$\begin{aligned} &\|[\tilde{u}, x_{-1}; \mathcal{K}]^{-1}([y^*, x^*; \mathcal{K}] - [\tilde{u}, x_{-1}; \mathcal{K}])\| \leq \lambda\beta M(\widetilde{A} + \widetilde{L}(\|\tilde{u} - y^*\| + \|x_{-1} - x^*\|)) \\ &\leq \lambda\beta M(\widetilde{A} + \widetilde{L}(2R + \alpha)) < \widetilde{h}(R) < 1, \end{aligned} \tag{3.4}$$

hence, by Banach Lemma the operator, $[y^*, x^*; \mathcal{K}]^{-1}$ exists and as

$$[y^*, x^*; \mathcal{K}](y^* - x^*) = \mathcal{K}(y^*) - \mathcal{K}(x^*) = 0,$$

then $y^* = x^*$. $\square$

### 3.2 An improvement of the uniqueness result

Here, we improve the uniqueness result that we have just proved by taking into account the uniqueness result of the Theorem 2 and inequality (3.4).

**Theorem 3.** *Under the conditions of Theorem 2, we assume that there exists* $R_1 \geq R$ *such that*

$$\lambda\beta M(\widetilde{A} + \widetilde{L}(R_1 + R + \alpha)) < 1,$$

*then,* $x^*$ *is the unique solution of equation* $\mathcal{K}(x) = 0$ *in* $\overline{B(\tilde{u}, R_1)} \cap \Omega$.

*Proof.* Let $y^*$ be another solution of equation $\mathcal{K}(x) = 0$ in $\overline{B(\tilde{u}, R_1)} \cap \Omega$. As in (3.4), we obtain

$$\begin{aligned} &\|[\tilde{u}, x_{-1}; \mathcal{K}]^{-1}([y^*, x^*; \mathcal{K}] - [\tilde{u}, x_{-1}; \mathcal{K}])\| \leq \lambda\beta M(\widetilde{A} + \widetilde{L}(\|\tilde{u} - y^*\| \\ &\quad + \|x_{-1} - x^*\|)) \leq \lambda\beta M(\widetilde{A} + \widetilde{L}(R_1 + \alpha + R)) < 1, \end{aligned}$$

and, as in Theorem 2, it follows that $y^* = x^*$. $\square$

### 3.3 Fixed-point type result for integral equation (1.1)

As we indicated in the Introduction, a solution of the equation $\mathcal{K}(x) = 0$ is a fixed point of the operator $\mathcal{H}$ and, therefore, a solution of the integral equation (1.1). Bearing this in mind, we can express the Theorem 2 as a fixed-point type result, with existence and uniqueness of solution for the integral equation (1.1), in the following form:

**Theorem 4.** *Let us consider the operator* $\mathcal{H}$ *given in* (1.2) *and assume conditions* ***(I)*** – ***(IV)*** *are verified. Then, there exists a unique fixed point of* $\mathcal{H}$ *in* $\overline{B(\tilde{u}, R)}$. *Moreover, the sequence*

$$\begin{cases} x_0, x_{-1} \text{ given in } \Omega, \ \mu \in [0,1], \\ y_n = (1-\mu)x_n + \mu x_{n-1}, \\ z_n = (1+\mu)x_n - \mu x_{n-1}, \\ x_{n+1} = x_n - [y_n, z_n; \mathcal{K}]^{-1} \mathcal{K}(x_n), \quad n \geq 0 \end{cases}$$

*converges to the fixed point of* $\mathcal{H}$ *for any initial point* $x_0$ *in* $\overline{B(\tilde{u},R)}$*, with* $x_0 \neq \tilde{u}$*, where the operator* $\mathcal{K}(x) = x - \mathcal{H}(x)$*.*

As we can see in the previous result, we have achieved our main objective in this work, obtaining a fixed-point-type result for operator $\mathcal{H}$. In addition, we have located a domain $\overline{B(\tilde{u},R)}$ in which there is a unique fixed point and iterative process (2.1) converges globally to said fixed point. Therefore, we have obtained a result of the existence and uniqueness of the solution for the integral equation (1.1). Moreover, we have established a procedure for its approximation with quadratic convergence.

## 4 Numerical experiments

Next, we present two numerical examples where we illustrate all the above results. In the first one, as we have previously indicated, the max-norm has been considered for setting the global convergence study. Second example is devoted to compare a family of secant-type iterative processes with the uniparametric family (2.1).

*Example 1.* We consider a nonlinear integral equation of Fredholm, which can be used to describe applied problems in the fields of electro-magnetics, fluid dynamics, in the kinetic theory of gases and, in general, in the reformulation of boundary value problems. So, we consider the equation of the form (1.1), given by

$$x(t) = g(t) + \beta \int_0^1 ts \left(x(s)^3 + |x(s)|\right) ds, \quad 0 \le t \le 1. \tag{4.1}$$

Notice that, $G(s,t) = st$ and $\mathcal{P}(x)(s) = x(s)^3 + |x(s)|$. We will consider $g(t) = (1 - \frac{11\beta}{80})t - \frac{1}{2}$, so that $x^*(t) = t - \frac{1}{2}$ is a solution of this integral equation.

Next, we obtain the bounds included in the assumptions **(I)**–**(IV)** for obtaining the results of Theorem 4. So, condition **(I)** gives us the following:

$$\begin{aligned}\|[u,v;\mathcal{P}] - [x,y;\mathcal{P}]\| &\le \left\|\frac{u^3 - v^3}{u - v} + \frac{|u| - |v|}{u - v} - \frac{x^3 - y^3}{x - y} + \frac{|x| - |y|}{x - y}\right\| \\ &\le A + L(\|u - x\| + \|v - y\|),\end{aligned}$$

for pairs of distinct points $(u,v),(x,y) \in \Omega \times \Omega \subseteq \mathcal{C}([b,c]) \times \mathcal{C}([b,c])$, with $A = 2$, $L = 3r$ and $\Omega = B(0,r)$, moreover we take $\widetilde{A} = A$ and $\widetilde{L} = L$.

For the divided difference for operator $\mathcal{K}$ is easy to obtain that:

$$\|[u,v;\mathcal{K}] - [x,y;\mathcal{K}]\| \le \beta M(2 + 3r\left(\|u - x\| + \|v - y\|\right)$$

with $M = \| \int_b^c stds\| = \frac{1}{2}$. Now, by taking $r = 1$, $\beta = 1/10$, $\tilde{u} = \frac{1}{2}$, $x_{-1} = 1/4$, $x_0 = 1/3 \in \Omega = B(0,1)$, we construct the auxiliary functions given in (3.1), with $\alpha = \frac{1}{4}$, and $\theta = 0.5625$. First equation gives us the value of $R$. We can see in Table 1 this radius of the global convergence ball for different values of parameter $\mu$. We observe that, if we take values of $\mu$ smaller, we get a better location of the fixed point in the global convergence balls obtained.

**Table 1.** Radii of existence and uniqueness of the global convergence balls for different values of $\mu$.

| $\mu$ | $R$ | $R_1$ |
|---|---|---|
| 0.2 | B(1/2, 0.088839) | $B(1/2, 4.327827) \cap B(0, 1)$ |
| 0.4 | B(1/2,0.092256) | $B(1/2, 4.324411) \cap B(0, 1)$ |
| 0.6 | B(1/2, 0.096332) | $B(1/2, 4.320333) \cap B(0, 1)$ |
| 0.8 | B(1/2, 0.101376) | $B(1/2, 4.315290) \cap B(0, 1)$ |
| 1.0 | B(1/2, 0.107986) | $B(1/2, 4.308679) \cap B(0, 1)$ |

Last column in this table is due to the uniqueness domain, solving equation given in Theorem 3.

Now, in order to solve the Equation (4.1) we approximate the integral by Simpson quadrature formula with $n$ subintervals $h = 1/n$ the corresponding weights $w = (1, 4, 2, 4, \ldots, 2, 4, 1)$ and nodes $s_j = hj, j = 0, 1, 2, \ldots, n$, and giving to $t$ the values $s_j$ with $j = 0, 1, \ldots, n$, the discretization of the problem gives us the following nonlinear system:

$$x_j = g_j + \frac{h}{3}\beta s_j \sum_{i=1}^{n} w_i s_i (x_i^3 + |x_i|) \quad j = 0, 1, 2, \ldots, n, \tag{4.2}$$

with $x_j = x(s_j)$ and $g_j = g(s_j)$, with $j = 0, 1, ..., n$.

Now, the system (4.2) can be written as

$$\mathcal{K}(x) \equiv \mathbf{x} - \mathbf{g} - \frac{\beta h}{3}\bar{s}\bar{w}\bar{s}\mathcal{P}(x) = 0,\ \mathcal{K}(x) : \mathbb{R}^n \to \mathbb{R}^n,\ \mathcal{K} = (K_1, K_2, \ldots, K_n), \tag{4.3}$$

where $\mathbf{x} = (x_0, x_1, x_2, \ldots, x_n)^T$, $\mathbf{g} = (g_0, g_1, g_2, \ldots, g_n)^T$ , $\bar{s} = diag(s)$, $\bar{w}$ is a square matrix of dimension $n + 1$ with all row equal to $w$ and

$$\mathcal{P}(x) = \left((x_0^3 + |x_0|), (x_1^3 + |x_1|), \ldots, (x_n^3 + |x_n|)\right)^T ,$$

so that $\mathcal{K}$ is nonlinear and nondifferentiable. Now, by taking into account (4.3) we note that the first order divided difference verifies:

$$[x, y; \mathcal{P}] = diag(x_i^2 + x_i y_i + y_i^2 + \frac{|x_i| - |y_i|}{x_i - y_i}) \quad i = 0, ..., n.$$

Finally, taking into account the study carried out previously in the infinite-dimensional case, by taking $n = 10$, with auxiliary point $\widetilde{\mathbf{u}} = \frac{1}{3}(1, ..., 1)^T$ and starting points $\mathbf{x_{-1}} = \frac{1}{2}(1, .., 1, 1)^T$ and $x_0 = \frac{1}{4}(1, ..., 1)^T$, we program the iterative schemes given in (2.1) in Matlab 20 by using variable precision arithmetic with 100 digits, using as stopping criteria $\|\mathbf{x_{n+1}} - \mathbf{x_n}\| < 10^{-30}$ and with the starting point $\mathbf{x_0}$ and $\mathbf{x_{-1}}$ mentioned above. Then, for different values of $\mu$ we obtain the approximated solution to the problem, we appreciate in Table 2 the number of iterations, the distance between the last two iterates, the norm of the $\mathcal{K}$ operator at the approximation to the solution and in the last row we have the approximated computational convergence order, defined in [11].

We can check in Table 2 the results showing that the behavior of the introduced method is always better that the initial Kurchatov's iterative method ($\mu = 1$) and we point out that Newton's method ($\mu = 0$) can not be applied for non differentiable operators.

**Table 2.** Numerical results with different values of parameter $\mu$.

| Method | (2.1) $\mu = 0.2$ | (2.1) $\mu = 0.4$ | (2.1) $\mu = 0.6$ | (2.1) $\mu = 0.8$ | Kurchatov $\mu = 1$ |
|---|---|---|---|---|---|
| iter | 6 | 6 | 6 | 6 | 6 |
| $\|\mathbf{x_{n+1}} - \mathbf{x_n}\|$ | 2.0710e-34 | 6.3649e-33 | 2.116e-32 | 5.987e-32 | 1.5443e-31 |
| $\|\mathcal{K}(\mathbf{x_{n+1}})\|$ | 1.9072e-69 | 1.0186e-66 | 1.0661e-65 | 8.3701e-65 | 5.515e-64 |
| ACOC | 2.0310 | 1.9184 | 1.8722 | 1.8406 | 1.8159 |

*Example 2.* We consider the boundary value problem given by

$$\left\{ \begin{array}{l} \dfrac{d^2x(s)}{ds^2} + \dfrac{x(s)^2 + |x(s)|}{4} = 0, \\ x(0) = 1/4 \text{ and } x(1) = 1/4. \end{array} \right. \tag{4.4}$$

It is known [20], that solving the previous boundary value problem, given by (4.4), is equivalent to solving a Fredholm integral equation of the form:

$$x(s) = \frac{1}{4} + \int_0^1 G(s,t)H(x(t))\,dt,$$

where the kernel $G$ is the Green function in $[0,1] \times [0,1]$:

$$G(s,t) = \left\{ \begin{array}{ll} (1-s)t, & t \le s, \\ s(1-t), & s \le t. \end{array} \right.$$

So, we take the Fredholm nonlinear integral equation given by

$$[\mathcal{H}(x)](s) = x(s) - \frac{1}{4} - \frac{1}{4}\int_0^1 G(s,t)(x(t)^2 + |x(t)|),\,dt, \quad s \in [0,1],$$

where $G$ is the Green's function, and $x$ is the solution to be obtained.

Next, we use the quadrature of Gauss-Legendre for discretizing the problem, obtaining the following nonlinear system:

$$x_j = \frac{1}{4} + \frac{1}{4}\sum_{i=1}^{n} p_{ji}\left(x_i^2 + |x_i|\right) \quad j = 1,2,\ldots,n, \tag{4.5}$$

with the corresponding weighs $q_j$ and nodes $t_j, j = 1,2,\ldots,n$,

$$p_{ij} = q_i G(t_j,t_i) = \left\{ \begin{array}{ll} q_i(1-t_j)t_i, & i \le j, \\ q_i(1-t_i)t_j, & i > j, \end{array} \right.$$

where $x_j = x(t_j)$, with $j = 1, \ldots, n$.

That is, the nonlinear system (4.5) in $\mathbb{R}^n$ is expressed as follows:

$$H(\mathbf{x}) \equiv \mathbf{x} - (1/4, \ldots, 1/4) - P\,\bar{\mathbf{x}} = 0, \quad H : \mathbb{R}^n \to \mathbb{R}^n, \quad H = (H_1, H_2, \ldots, H_n),$$

where the matrix $P = (p_{ij})_{i,j=1}^n$ and $\bar{\mathbf{x}} = \left(x_1^2 + |x_1|, x_2^2 + |x_2|, \ldots, x_n^2 + |w_n|\right)^T$.

In order to compare numerical results obtained by our family (2.1), we use the family of secant-type iterative schemes given in [12]:

$$\begin{cases} w_{-1}, w_0 \text{ given in } \Omega, \quad \lambda \in [0,1], \\ u_n = \lambda w_n + (1-\lambda) w_{n-1}, \quad n \geq 0, \\ w_{n+1} = w_n - [u_n, w_n; H]^{-1} H(w_n), \end{cases} \tag{4.6}$$

with $\lambda \in [0,1]$. Notice that, if $\lambda = 0$ we have the secant method while if $\lambda = 1$ and $H$ is differentiable one has Newton's method.

**Table 3.** Numerical results with different values of parameter $\mu$ for (2.1).

| Method | Kurchatov $\mu = 1$ | (2.1) $\mu = 0.1$ | (2.1) $\mu = 0.05$ |
|---|---|---|---|
| k | 6 | 5 | 5 |
| $\lVert\mathbf{x_{n+1}} - \mathbf{x_n}\rVert$ | 1.9789e-51 | 1.1557e-30 | 1.1557e-30 |
| $\lVert H(\mathbf{x_{n+1}})\rVert$ | 6.2180e-59 | 6.6046e-59 | 6.6046e-59 |

**Table 4.** Numerical results with different values of parameter $\lambda$ for (4.6).

| Method | Secant like (4.6) $\lambda = 0$ | Secant like (4.6) $\lambda = 0.5$ | Secant like (4.6) $\lambda = 0.8$ |
|---|---|---|---|
| k | 7 | 6 | 6 |
| $\lVert\mathbf{x_{n+1}} - \mathbf{x_n}\rVert$ | 2.5456e-37 | 1.1028e-27 | 4.5699e-32 |
| $\lVert H(\mathbf{x_{n+1}})\rVert$ | 5.2687e-59 | 2.1741e-45 | 1.0203e-52 |

The results in Tables 3 and 4 respectively obtained for (2.1) and (4.6) have been obtained with the same digits and stopping criteria conditions that in Example 1, starting with $\mathbf{x_0} = (\frac{1}{5}, ..., \frac{1}{5})^T$ and $\mathbf{x_{-1}} = (\frac{1}{2}, ..., \frac{1}{2})^T$. The number of iterations needed $k$ and the asymptotic error show the competitiveness of our family (2.1).

## 5 Conclusions

In this work, we focus on integral equations of Fredholm-type of the second kind, that is with Nemytskii operator non differentiable. We obtain a restricted fixed point result for the nonlinear operator that defines the problem which does not use successive approximations of linear convergence, instead of it we use iterative schemes of second order of convergence. The global convergence of these methods is obtained by using auxiliary points. Finally, we deal with some numerical experiments where we illustrate the obtained theoretical results.

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