ISSN: 1392-6292/ eISSN: 1648-3510

VILNIUS TECH Vilnius Gediminas Technical University

MATHEMATICAL MODELLING and ANALYSIS

2025
Volume 30
Issue 1
Pages 142–158
https://doi.org/10.3846/mma.2025.21939

Approximation of analytic functions by generalized shifts of the Lerch zeta-function

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Article History:
- received July 31, 2024
- revised September 23, 2024
- accepted October 4, 2024

Abstract. In the paper, we approximate analytic functions by generalized shifts $L(\lambda,\alpha,s+ig(\tau))$, $s=\sigma+it$, of the Lerch zeta-function, where g is a certain increasing to $+\infty$ real function having a monotonic derivative. We prove that, for arbitrary parameters λ and α, there exists a closed set $\mathfrak{F}_{\lambda,\alpha}$ of analytic functions defined in the strip $1/2<\sigma<1$ which functions are approximated by the above shifts. If the set of logarithms $\log(m+\alpha)$, $m\in\mathbb{N}_0$, is linearly independent over the field of rational numbers, then the set $\mathfrak{F}_{\lambda,\alpha}$ coincides with the set of all analytic functions in that strip.

Keywords: Lerch zeta-function; Mergelyan theorem; space of analytic functions universality; weak convergence.

AMS Subject Classification: 11M35.

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1 Introduction

Let $0<\alpha\leqslant 1$ and $\lambda\in\mathbb{R}$ be two fixed parameters, and $s=\sigma+it$, $\sigma,t\in\mathbb{R}$, $i=\sqrt{-1}$, a complex variable. The Lerch zeta-function $L(\lambda,\alpha,s)$ was introduced in [24], and is defined, for $\sigma>1$, by the Dirichlet series

$$L(\lambda,\alpha,s)=\sum_{m=0}^{\infty}\frac{\mathrm{e}^{2\pi i\lambda m}}{(m+\alpha)^s}.$$

For $\lambda\notin\mathbb{Z}$, the function $L(\lambda,\alpha,s)$ has analytic continuation to the whole complex plane, while, for $\lambda\in\mathbb{Z}$, coincides with the Hurwitz zeta-function

$$\zeta(s,\alpha)=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^s},\quad \sigma>1,$$

which is meromorphic on $\mathbb{C}$ with unique simple pole $s = 1$ with residue 1. Moreover, with $k \in \mathbb{Z}$

$$L(k, 1, s) = \zeta(s),$$

where $\zeta(s)$ is the famous Riemann zeta-function. These remarks show that the Lerch zeta-function is a generalization of the classical zeta-functions $\zeta(s, \alpha)$ and $\zeta(s)$.

The function $L(\lambda, \alpha, s)$, as $\zeta(s, \alpha)$ and $\zeta(s)$, satisfies the functional equation connecting the variables s and $1 - s$. Let, as usual, $\Gamma(s)$ denote the Euler gamma-function. Then, for $0 < \lambda < 1$ and all $s \in \mathbb{C}$,

$$\begin{aligned} L(\lambda, \alpha, 1-s) =& (2\pi)^{-s}\Gamma(s)\left(\exp\left\{\frac{\pi i s}{2} - 2\pi i \alpha\lambda\right\} L(-\alpha, \lambda, s)\right. \\ &\left. + \exp\left\{-\frac{\pi i s}{2} + 2\pi i \alpha(1-\lambda)\right\} L(\alpha, 1-\lambda, s)\right). \qquad (1.1) \end{aligned}$$

The latter equation for the first time was proved in [24]. Later several authors, among them Berndt [4], Apostol [1, 2], Oberhettinger [28], Mikolás [27] gave another proofs of (1.1). In general, the function $L(\lambda, \alpha, s)$ is an interesting analytic object governed by two parameters, and has wide applications in special function theory and algebraic number theory. The analytic theory, including (1.1), of the Lerch zeta-function can be found in [17].

Our purpose is approximation of analytic functions by shifts $L(\lambda, \alpha, s+i\tau)$, $\tau \in \mathbb{R}$. The problem of approximation of analytic functions by shifts of zeta-functions comes back to Voronin who discovered in [32] the universality of the Riemann zeta-function. Let $0 < r < 1/4$, $f(s)$ be a non-vanishing analytic function on $|s| \leqslant r$, and analytic in $|s| < r$. Then, Voronin proved [32] that, for every $\varepsilon > 0$, there is a number $\tau = \tau(\varepsilon) \in \mathbb{R}$ such that

$$\max_{|s|\leqslant r} \left| f(s) - \zeta\left(s + 3/4 + i\tau\right)\right| < \varepsilon.$$

A bit later, Voronin theorem was improved in [3, 7, 12, 17, 31], additionally see [9], and extended for other zeta-functions. Its last version says that every non-vanishing analytic on the strip $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ function can be approximated with a given accuracy by shifts $\zeta(s + i\tau)$. The set of such shifts is infinite, it has a positive lower density. Voronin's theorem has various theoretical and practical applications (functional independence, zero-distribution, moment problem, quantum mechanics), see a survey paper [25]. The importance of universality phenomenon of zeta-functions stimulates investigations in the field including the extension of the set of universal functions. The Lerch zeta-function which analytic properties depend on arithmetic of two parameters is suitable object for development of universality.

The first result on approximation of analytic functions by shifts $L(\lambda, \alpha, s + i\tau)$, $\tau \in \mathbb{R}$, has been obtained in [13]. Let $\mathcal{K}$ be the class of compact subsets of the strip D with connected complements, and $\mathcal{H}(K)$ with $K \in \mathcal{K}$ the class of continuous functions on K that are analytic in the interior of K. Denote by

$\mathfrak{L}A$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Remind that a number α is called transcendental if it is not a root of any non-trivial polynomial with rational coefficients. In the oposite case, α is algebraic. Then, the main result of [13] is stated as follows.

Theorem 1. *Suppose that the parameter α is transcendental. Let $K \in \mathcal{K}$ and $f(s) \in \mathcal{H}(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mathfrak{L}\left\{\tau \in [0,T] : \sup_{s\in K} |f(s) - L(\lambda,\alpha,s+i\tau)| < \varepsilon\right\} > 0. \tag{1.2}$$

Theorem 1 implies that the set of shifts $L(\lambda,\alpha,s+i\tau)$ approximating a given function $f(s) \in \mathcal{H}(K)$ is infinite. On the other hand, Theorem 1 is non-effective because any concrete shift $L(\lambda,\alpha,s+i\tau)$ approximating $f(s)$ is not known.

The parameter λ in Theorem 1 is an arbitrary real number. By the way, it is sufficient to limit investigation for $0 < \lambda \leqslant 1$ due to periodicity of the coefficients $\mathrm{e}^{2\pi i\lambda m}$.

We notice that the transcendence of the parameter α in Theorem 1 can be replaced by a linear independence over the field of rational numbers $\mathbb{Q}$ for the set

$$V(\alpha) \stackrel{\text{def}}{=} \{\log(m+\alpha) : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}.$$

Suppose that the parameter λ is rational, say, $\lambda = a/b$, $a < b$, $(a,b) = 1$, and α is arbitrary. Then, for $\sigma > 1$, we have

$$L(\lambda,\alpha,s) = \frac{1}{b^s}\sum_{k=0}^{b-1} \mathrm{e}^{2\pi i\lambda k} \sum_{m=0}^{\infty} \frac{1}{(m+(k+\alpha)/b)^s} = \frac{1}{b^s}\sum_{k=0}^{b-1} \mathrm{e}^{2\pi i\lambda k}\zeta\left(s, \frac{k+\alpha}{b}\right).$$

Hence, the consideration of the function $L(\lambda,\alpha,s)$ reduces to joint that of Hurwitz zeta-functions, and this was realized in [18]. However, if λ is not rational, then the latter way does not work. Additionally, we do not know any universality result for $L(\lambda,\alpha,s)$ with algebraic irrational parameter α. Taking into account this problem, in [15] a result confirming good approximation properties of the function $L(\lambda,\alpha,s)$ with arbitrary parameters λ and α was proposed. Denote by $\mathcal{H}(D)$ the space of analytic on D functions endowed with the topology of uniform convergence on compacta. Then, the following theorem was given in [15].

Theorem 2. *Suppose that the parameters $0 < \lambda \leqslant 1$ and $0 < \alpha \leqslant 1$ are arbitrary. Then, there exists a non-empty closed set $\mathfrak{F}_{\lambda,\alpha} \subset \mathcal{H}(D)$ such that, for every compact set $K \subset D$, $f(s) \in \mathfrak{F}_{\lambda,\alpha}$ and $\varepsilon > 0$, inequality (1.2) is true. Moreover, the limit*

$$\lim_{T\to\infty} \frac{1}{T}\mathfrak{L}\left\{\tau \in [0,T] : \sup_{s\in K} |f(s) - L(\lambda,\alpha,s+i\tau)| < \varepsilon\right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

The purpose of the present paper is an extension of Theorems 1 and 2 for generalized shifts $L(\lambda,\alpha,s+ig(\tau))$ with a certain class of real functions $g(\tau)$.

This is inspired by the paper [29]. Researches of such a kind were continued in [6, 8, 10, 11, 14, 16, 19, 20, 21, 22, 23, 30].

We say that a real-valued function $g(\tau)$ belongs to the class $U(T_0)$ if:
1° $g(\tau)$ is defined for $\tau \geqslant 0$, and $g(\tau) \to \infty$ as $\tau \to \infty$;
2° $g(\tau)$, for $\tau \geqslant T_0$, $T_0 > 0$, has a monotonic derivative;
3° The estimate $g(2\tau) \ll \tau \min(g'(\tau), g'(2\tau))$, as $\tau \to \infty$, is valid.

Here and in what follows, the notation $x \ll_\theta y$, $x \in \mathbb{C}$, $y > 0$, is the synonym of $x = O(y)$ with implied constant depending on the parameter θ.

Now, we state the main results of the paper.

Theorem 3. *Suppose that the set $V(\alpha)$ is linearly independent over $\mathbb{Q}$, and $g(\tau) \in U(T_0)$. Let $K \in \mathcal{K}$ and $f(s) \in \mathcal{H}(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mathfrak{L}\Big\{\tau \in [0,T] : \sup_{s\in K} |f(s) - L(\lambda,\alpha,s+ig(\tau))| < \varepsilon\Big\} > 0. \tag{1.3}$$

Moreover, the limit

$$\lim_{T\to\infty} \frac{1}{T}\mathfrak{L}\Big\{\tau \in [0,T] : \sup_{s\in K} |f(s) - L(\lambda,\alpha,s+ig(\tau))| < \varepsilon\Big\} \tag{1.4}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

Theorem 4. *Suppose that the parameters $0 < \lambda \leqslant 1$ and $0 < \alpha \leqslant 1$ are arbitrary, and $g(\tau) \in U(T_0)$. Then, there exists a non-empty closed set $\mathfrak{F}_{\lambda,\alpha,g} \subset \mathcal{H}(D)$ such that, for every compact set $K \subset D$, $f(s) \in \mathfrak{F}_{\lambda,\alpha,g}$ and $\varepsilon > 0$, inequality (1.3) is true. Moreover, the limit (1.4) exists and is positive for all but at most countably many $\varepsilon > 0$.*

For example, every polynomial with positive main coefficient is an element of $U(1)$. The Gram function is the solution $g(\tau)$ of the equation $\varphi(t) = (\tau - 1)\pi$, $\tau \geqslant 0$, where $\varphi(t)$ is the increment of the argument of $\pi^{-s/2}\Gamma(s/2)$ along straight line connecting by points $1/2$ and $1/2 + it$. Then $g(\tau)$ as well as belongs to $U(2)$.

Proofs of Theorems 3 and 4 are based on probabilistic limit theorems on weak convergence of probability measures in the space $\mathcal{H}(D)$.

2 Infinite-dimensional torus

We start with a limit lemma on the torus $\Omega = \prod_{m\in\mathbb{N}_0}\{s \in \mathbb{C} : |s| = 1\}$, which is the set of all functions defined on $\mathbb{N}_0$ and taking values on the unit circle. The infinite-dimensional torus Ω with the pointwise multiplication and product topology can be considered as topological compact group. Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of a space $\mathbb{X}$, and by $\omega = (\omega(m) : m \in \mathbb{N}_0)$ elements of Ω.

For $A \in \mathcal{B}(\Omega)$, define

$$P_T^{\Omega}(A) = \frac{1}{T}\mathfrak{L}\Big\{\tau \in [0,T] : \Big((m+\alpha)^{-ig(\tau)} : m \in \mathbb{N}_0\Big) \in A\Big\}.$$

Lemma 1. *Suppose that* $g(\tau) \in U(T_0)$*. Then* P_T^Ω *converges weakly to a certain probability measure* P^Ω *on* $(\Omega, \mathcal{B}(\Omega))$ *as* $T \to \infty$*.*

Proof. Since Ω is a compact group, it is convenient to use the Fourier transforms. Thus, let $F_T^\Omega(\underline{k})$, $\underline{k} = (k_m : k_m \in \mathbb{Z},\, m \in \mathbb{N}_0)$, be the Fourier transform of P_T^Ω, i.e.,

$$F_T^\Omega(\underline{k}) = \int_\Omega \prod_{m\in\mathbb{N}_0}^{*} \omega^{k_m}(m)\, \mathrm{d}P_T^\Omega,$$

where the sign $*$ indicates that only a finite number of integers k_m are not zeros. Therefore, by the definition of P_T^Ω,

$$\begin{aligned} F_T^\Omega(\underline{k}) &= \frac{1}{T}\int_0^T \prod_{m\in\mathbb{N}_0}^{*} (m+\alpha)^{-ik_m g(\tau)}\, \mathrm{d}\tau \\ &= \frac{1}{T}\int_0^T \exp\Big\{ -ig(\tau) \sum_{m\in\mathbb{N}_0}^{*} k_m \log(m+\alpha) \Big\}\, \mathrm{d}\tau. \end{aligned} \tag{2.1}$$

Let

$$\underline{k}_1 = \Big\{ \underline{k} : \sum_{m\in\mathbb{N}_0}^{*} k_m \log(m+\alpha) = 0 \Big\}, \quad \underline{k}_2 = \Big\{ \underline{k} : \sum_{m\in\mathbb{N}_0}^{*} k_m \log(m+\alpha) \neq 0 \Big\}.$$

Obviously, for $\underline{k} \in \underline{k}_1$,

$$F_T^\Omega(\underline{k}) = 1. \tag{2.2}$$

For brevity, let, for $\underline{k} \in \underline{k}_2$,

$$A(\tau, \underline{k}) = g(\tau) \sum_{m\in\mathbb{N}_0}^{*} k_m \log(m+\alpha).$$

Then, by (2.1), as $T \to \infty$,

$$\begin{aligned} \mathrm{Re} F_T^\Omega(\underline{k}) &= \frac{1}{T}\int_0^T \cos A(\tau, \underline{k})\, \mathrm{d}\tau = \frac{1}{T}\int_{T_0}^T \cos A(\tau, \underline{k})\, \mathrm{d}\tau + o(1) \\ &= \frac{1}{T}\int_{T_0}^T \frac{1}{A'(\tau, \underline{k})}\, \mathrm{d} \sin A(\tau, \underline{k}) + o(1). \end{aligned} \tag{2.3}$$

By the second mean value theorem, as $T \to \infty$,

$$\int_{T_0}^T \frac{1}{A'(\tau, \underline{k})}\, \mathrm{d}\sin A(\tau, \underline{k}) \ll \frac{1}{|A'(T_0, \underline{k})|} + \frac{1}{|A'(T, \underline{k})|} = o(T).$$

This and (2.3) show that, $\mathrm{Re} F_T^\Omega(\underline{k}) = o(1)$ as $T \to \infty$. Similarly, we obtain the estimate $\mathrm{Im} F_T^\Omega(\underline{k}) = o(1)$, as $T \to \infty$. Thus, for $\underline{k} \in \underline{k}_2$,

$$\lim_{T\to\infty} F_T^\Omega(\underline{k}) = 0.$$

This, together with (2.2), gives

$$\lim_{T\to\infty} F_T^{\Omega}(\underline{k}) = \begin{cases} 1, & \text{if } \underline{k} \in \underline{k}_1, \\ 0, & \text{if } \underline{k} \in \underline{k}_2. \end{cases}$$

Hence, the measure P_T^{Ω} converges weakly to the measure P^{Ω} defined by the Fourier transform

$$F^{\Omega}(\underline{k}) = \begin{cases} 1, & \text{if } \underline{k} \in \underline{k}_1, \\ 0, & \text{if } \underline{k} \in \underline{k}_2. \end{cases} \tag{2.4}$$

□

Lemma 2. *Suppose that $g(\tau) \in U(T_0)$ and the set $V(\alpha)$ is linearly independent over $\mathbb{Q}$. Then P_T^{Ω} converges weakly to the Haar measure m_H as $T \to \infty$.*

Proof. Since the set $V(\alpha)$ is linearly independent over $\mathbb{Q}$,

$$\sum_{m\in\mathbb{N}_0}^{*} k_m \log(m+\alpha) = 0$$

if and only if $\underline{k} = \underline{0}$, where $\underline{0} = (0, 0, \dots)$. Therefore, in view of (2.4), P_T^{Ω}, as $T \to \infty$, converges weakly to the measure given by the Fourier transform

$$F^{\Omega}(\underline{k}) = \begin{cases} 1, & \text{if } \underline{k} = \underline{0}, \\ 0, & \text{otherwise}, \end{cases}$$

i.e., to the Haar measure m_H. □

3 Absolutely convergent series

The weak convergence of the measure P_T^{Ω} allows to consider that for measures defined by absolutely convergent Dirichlet series.

Let $\eta > 1/2$ be a fixed number, and, for $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$,

$$u_n(m,\alpha) = \exp\left\{ -\left(\frac{m+\alpha}{n}\right)^{\eta} \right\}.$$

Consider a Dirichlet series for $L(\lambda, \alpha, s)$ twisted by the coefficients $u_n(m, \alpha)$, i.e.,

$$L_n(\lambda,\alpha,s) \stackrel{\text{def}}{=} \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m} u_n(m,\alpha)}{(m+\alpha)^s}.$$

Since the coefficients $u_n(m, \alpha)$ with respect to m tend to zero exponentially, the series for $L_n(\lambda, \alpha, s)$ converges absolutely in every half-plane $\sigma > \sigma_0$. For $A \in \mathcal{B}(\mathcal{H}(D))$, define

$$P_{T,n,\lambda,\alpha}(A) = \frac{1}{T}\mathfrak{L}\left\{\tau \in [0,T] : L_n(\lambda,\alpha,s+ig(\tau)) \in A\right\}.$$

Lemma 3. *Suppose that $g(\tau) \in U(T_0)$. Then, $P_{T,n,\lambda,\alpha}$ converges weakly to a certain probability measure $P_{n,\lambda,\alpha}$ as $T \to \infty$.*

Proof. We apply a preservation phenomenon of weak convergence under continuous mappings, see, for example, Section 5.1 of [5]. For $\omega \in \Omega$, set

$$L_n(\lambda, \alpha, \omega, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}\omega(m)u_n(m, \alpha)}{(m+\alpha)^s}.$$

Since $|\omega(m)| = 1$, the latter series, as for $L_n(\lambda, \alpha, s)$, converges absolutely for $\sigma > \sigma_0$. Define a mapping $h_{n,\lambda,\alpha} : \Omega \to \mathcal{H}(D)$ by $h_{n,\lambda,\alpha}(\omega) = L_n(\lambda, \alpha, \omega, s)$.

The torus Ω is endowed with the product topology, therefore, the absolute convergence of the series for $L_n(\lambda, \alpha, \omega, s)$ ensures a continuity for $h_{n,\lambda,\alpha}$. Hence, the mapping $h_{n,\lambda,\alpha}$ is $(\mathcal{B}(\Omega), \mathcal{B}(\mathcal{H}(D)))$-measurable. Thus, every probability measure P on Ω induces the unique probability measure $Ph^{-1}_{n,\lambda,\alpha}$ on $\mathcal{B}(\mathcal{H}(D))$ given by

$$Ph^{-1}_{n,\lambda,\alpha}(A) = P(h^{-1}_{n,\lambda,\alpha}A), \quad A \in \mathcal{B}(\mathcal{H}(D)).$$

Moreover, if P_N, as $N \to \infty$, converges to P, then $P_N h^{-1}_{n,\lambda,\alpha}$ converges weakly to $Ph_{n,\lambda,\alpha}$ as $N \to \infty$. All these remarks are given in [5].

Now, apply the above theory for measures $P_{T,n,\lambda,\alpha}$, P^{Ω}_T and P^{Ω}. The definitions of the latter measures and $h_{n,\lambda,\alpha}$ yield, for $A \in \mathcal{B}(\mathcal{H}(D))$,

$$\begin{aligned} P_{T,n,\lambda,\alpha}(A) &= \frac{1}{T}\mathfrak{L}\Big\{\tau \in [0,T] : h_{n,\lambda,\alpha}\Big((m+\alpha)^{-ig(\tau)} : m \in \mathbb{N}_0\Big) \in A\Big\} \\ &= \frac{1}{T}\mathfrak{L}\Big\{\tau \in [0,T] : \Big((m+\alpha)^{-ig(\tau)} : m \in \mathbb{N}_0\Big) \in h^{-1}_{n,\lambda,\alpha}A\Big\} = P^{\Omega}_T(h^{-1}_{n,\lambda,\alpha}A). \end{aligned}$$

Since A is arbitrary, we have $P_{T,n,\lambda,\alpha} = P^{\Omega}_T h^{-1}_{n,\lambda,\alpha}$. This equality, continuity of $h_{n,\lambda,\alpha}$, property of preservation of weak convergence under continuous mappings and Lemma 1 show that $P_{T,n,\lambda,\alpha}$ converges weakly to $P_{n,\lambda,\alpha}$ as $T \to \infty$, where $P_{n,\lambda,\alpha}$ is a probability measure on $(\mathcal{H}(D), \mathcal{B}(\mathcal{H}(D)))$ given by

$$P_{n,\lambda,\alpha} = P^{\Omega}h^{-1}_{n,\lambda,\alpha}. \tag{3.1}$$

□

Lemma 4. *Suppose that $g(\tau) \in U(T_0)$ and the set $V(\alpha)$ is linearly independent over $\mathbb{Q}$. Then, $P_{T,n,\lambda,\alpha}$ converges weakly to the measure $m_H h^{-1}_{n,\lambda,\alpha}$ as $T \to \infty$.*

Proof. It suffices to use (3.1) and apply Lemma 2. □

4 Approximation result

To pass from $L_n(\lambda, \alpha, s)$ to $L(\lambda, \alpha, s)$, some distance estimates are needed.

For convenience we remind a metric in the space $\mathcal{H}(D)$ which induces the topology of uniform convergence on compacta. Let $\{K_r : r \in \mathbb{N}\} \subset D$ be a sequence of compact subsets satisfying:

1° $K_r \subset K_{r+1}$, for $r \in \mathbb{N}$;
2° The strip D is the union of sets K_r;
3° Every compact set $K \subset D$ lies in some K_r.

Such a sequence $\{K_r\}$ exists, for example, we can take embedded rectangles.

For $f_1, f_2 \in \mathcal{H}(D)$, set

$$d(f_1, f_2) = \sum_{r=1}^{\infty} \frac{1}{2^r} \frac{\sup_{s\in K_r} |f_1(s) - f_2(s)|}{1 + \sup_{s\in K_r} |f_1(s) - f_2(s)|}.$$

Then, d is a metric in $\mathcal{H}(D)$ inducing its topology of uniform convergence on compacta.

Lemma 5. *Suppose that $g(\tau) \in U(T_0)$. Then, the equality*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T d\left(L(\lambda, \alpha, s + ig(\tau)), L_n(\lambda, \alpha, s + ig(\tau))\right) \mathrm{d}\tau = 0$$

holds.

Proof. The Mellin formula

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(z) b^{-z} \,\mathrm{d}z = \mathrm{e}^{-b}, \quad a, b > 0,$$

gives

$$\frac{1}{2\pi i} \int\limits_{\eta-i\infty}^{\eta+i\infty} \frac{1}{\eta} \Gamma\left(\frac{z}{\eta}\right) \left(\frac{m+\alpha}{n}\right)^{-z} \mathrm{d}z = \int\limits_{\eta-i\infty}^{\eta+i\infty} \Gamma\left(\frac{z}{\eta}\right) \left(\frac{m+\alpha}{n}\right)^{(-z/\eta)\eta} \mathrm{d}\left(\frac{z}{\eta}\right)$$
$$= \exp\left\{-\left((m+\alpha)/n\right)^{\eta}\right\}.$$

Therefore, for $\sigma > 1/2$,

$$\begin{aligned} L_n(\lambda, \alpha, s) &= \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i\lambda m}}{(m+\alpha)^s} \left(\frac{1}{2\pi i} \int_{\eta-i\infty}^{\eta+i\infty} \frac{1}{\eta} \Gamma\left(\frac{z}{\eta}\right) \left(\frac{m+\alpha}{n}\right)^{-z} \mathrm{d}z \right) \\ &= \frac{1}{2\pi i} \int_{\eta-i\infty}^{\eta+i\infty} \left(\sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i\lambda m}}{(m+\alpha)^{s+z}} \right) \left(\frac{1}{\eta} \Gamma\left(\frac{z}{\eta}\right) n^z \right) \mathrm{d}z \\ &= \frac{1}{2\pi i} \int_{\eta-i\infty}^{\eta+i\infty} L(\lambda, \alpha, s+z) a_n(z) \,\mathrm{d}z, \end{aligned} \tag{4.1}$$

where $a_n(s) = \frac{1}{\eta} \Gamma\left(\frac{s}{\eta}\right) n^s$. Let $K \subset D$ be a compact set. Then, there is $\delta > 0$ such that $1/2 + \delta \leqslant \sigma \leqslant 1 - \delta/2$ for $s = \sigma + it \in K$. Move the line of integration in (4.1) to the left. Let $\eta = 1/2 + \delta/2$ and $\eta_1 = 1/2 + \delta/2 - \sigma$. Then, $-1/2 + \delta \leqslant \eta_1 \leqslant -\delta$. Therefore, the integrand in (4.1) has a simple pole at the point $z = 0$ if $0 < \lambda < 1$, and, additionally, a simple pole at the point $z = 1 - s$ if $\lambda = 1$. Therefore, by the Cauchy theorem, (4.1) implies

$$L_n(\lambda, \alpha, s) - L(\lambda, \alpha, s) = \frac{1}{2\pi i} \int_{\eta_1 - i\infty}^{\eta_1 + i\infty} L(\lambda, \alpha, s + z) a_n(z) \, \mathrm{d}z + \begin{cases} a_n(1-s), & \text{if } \lambda = 1, \\ 0, & \text{if } 0 < \lambda < 1. \end{cases}$$

Hence,

$$\begin{aligned} &\sup_{s \in K} \left| L(\lambda, \alpha, s + ig(\tau)) - L_n(\lambda, \alpha, s + ig(\tau)) \right| \\ &\quad \ll \int_{-\infty}^{\infty} \left| L\left(\lambda, \alpha, \frac{1}{2} + \frac{\delta}{2} + iv + ig(\tau)\right) \right| \sup_{s \in K} \left| a_n\left(\frac{1}{2} + \frac{\delta}{2} - s + iv\right) \right| \mathrm{d}v \\ &\qquad + \sup_{s \in K} \left| a_n(1 - s - ig(\tau)) \right| \end{aligned}$$

after a shift $t + v \to v$. Therefore, the estimate

$$\Gamma(\sigma + it) \ll \exp\{-c|t|\}, \quad c > 0 \tag{4.2}$$

yields

$$\begin{aligned} &\frac{1}{T} \int_0^T \sup_{s \in K} \left| L(\lambda, \alpha, s + ig(\tau)) - L_n(\lambda, \alpha, s + ig(\tau)) \right| \mathrm{d}\tau \\ &\quad \ll \int_{-\infty}^{\infty} \left(\frac{1}{T} \int_0^T \left| L\Big(\lambda, \alpha, \frac{1}{2} + \frac{\delta}{2} + iv + ig(\tau)\Big) \right| \mathrm{d}\tau \right) \\ &\qquad \times \sup_{s \in K} \left| a_n\left(1/2 + \delta/2 - s + iv\right) \right| \mathrm{d}v + \frac{1}{T} \int_0^T \sup_{s \in K} \left| a_n(1 - s - ig(\tau)) \right| \mathrm{d}\tau \\ &\quad \stackrel{\text{def}}{=} I + \widehat{I}. \end{aligned} \tag{4.3}$$

The main problem in estimation of I is the estimate for

$$\begin{aligned} J &\stackrel{\text{def}}{=} \frac{1}{T} \int_0^T \left| L\left(\lambda, \alpha, 1/2 + \delta/2 + iv + ig(\tau)\right) \right| \mathrm{d}\tau \\ &\ll \left(\frac{1}{T} \int_0^T \left| L\left(\lambda, \alpha, 1/2 + \delta/2 + iv + ig(\tau)\right) \right|^2 \mathrm{d}\tau \right)^{1/2}, \end{aligned} \tag{4.4}$$

with all $v \in \mathbb{R}$. It is well known, see, for example, [17], that, for fixed σ, $1/2 < \sigma < 1$,

$$\int_0^T |L(\lambda, \alpha, \sigma + it)|^2 \, \mathrm{d}t \ll_{\lambda, \alpha, \sigma} T, \quad T \to \infty.$$

From this, for the same σ, we have

$$\int_{-T}^T |L(\lambda, \alpha, \sigma + it)|^2 \, \mathrm{d}t \ll_{\lambda, \alpha, \sigma} T. \tag{4.5}$$

Moreover, for $\sigma \geqslant 1/2$, the bound [17]

$$L(\lambda,\alpha,\sigma+it) \ll_{\lambda,\alpha,\sigma} (1+|t|)^{1/2} \tag{4.6}$$

is valid. For $V \geqslant T_0$, in view of (4.5) and properties of the class $U(T_0)$, we have

$$\begin{aligned}
&\int_V^{2V} \left| L\left(\lambda,\alpha,\frac{1}{2}+\frac{\delta}{2}+iv+ig(\tau)\right)\right| \mathrm{d}\tau \\
&\quad = \int_V^{2V} \left| L\left(\lambda,\alpha,\frac{1}{2}+\frac{\delta}{2}+iv+ig(\tau)\right)\right| \frac{\mathrm{d}g(\tau)}{g'(\tau)} \\
&\quad \ll \left(\frac{1}{g(V)}+\frac{1}{g(2V)}\right) \int_{-|v|-g'(V)}^{|v|+g'(2V)} \left| L\left(\lambda,\alpha,\frac{1}{2}+\frac{\delta}{2}+iu\right)\right|^2 \mathrm{d}u \\
&\quad \ll_{\lambda,\alpha,\delta} \frac{|v|+g(2V)}{\min(g'(V),g'(2V))} \ll_{\lambda,\alpha,\delta} \frac{|g(2V)(1+|v|)}{\min(g'(V),g'(2V))} \ll_{\lambda,\alpha,\delta} V(1+|v|).
\end{aligned}$$

Now, taking $V = T2^{-k}$, and summing over $k = 1,2,\ldots$, we obtain from this and (4.6)

$$J^2 \ll_{\lambda,\alpha,\delta} \frac{1}{T}\int_{T_0}^{T} \left| L\left(\lambda,\alpha,\frac{1}{2}+\frac{\delta}{2}+iv+ig(\tau)\right)\right|^2 \mathrm{d}\tau + (1+|v|) \ll_{\lambda,\alpha,\delta} 1+|v|. \tag{4.7}$$

By (4.2), for $s \in K$,

$$\begin{aligned}
a_n\left(\frac{1}{2}+\frac{\delta}{2}-s-iv\right) &\ll_\delta n^{1/2+\delta/2-\sigma} \exp\left\{-\frac{c}{\eta}|v-t|\right\} \\
&\ll_\delta n^{-\delta/2} \exp\{-c_1|v|\}, \quad c_1 > 0.
\end{aligned}$$

This, (4.7) and (4.3) yield

$$I \ll_{\lambda,\alpha,\delta,K} n^{-\delta/2} \int_{-\infty}^{\infty} (1+|v|)^{1/2} \exp\{-c_1|v|\}\,\mathrm{d}v \ll_{\lambda,\alpha,\delta,K} n^{-\delta/2}. \tag{4.8}$$

Similarly as above, for $s \in K$, we have

$$a_n(1-s-ig(\tau)) \ll n^{1-\sigma} \exp\left\{-\frac{c}{\eta}|t+g(\tau)|\right\} \ll_{\delta,K} n^{1/2-\delta} \exp\{-c_2 g(\tau)\}.$$

Therefore,

$$\begin{aligned}
\widehat{I} &\ll_{\delta,K} \frac{n^{1/2-\delta}}{T} \int_0^T \exp\{-c_2 g(\tau)\}\,\mathrm{d}\tau \\
&\ll_{\delta,K} \frac{n^{1/2-\delta}\log T}{T} + n^{1/2-\delta} \int_{\log T}^{T} \exp\{-c_2 g(\tau)\}\,\mathrm{d}\tau \\
&\ll_{\delta,K} n^{1/2-\delta}\frac{\log T}{T} + n^{1/2-\delta} \exp\{-c_2 g(\log T)\}.
\end{aligned}$$

This shows that $\lim_{T\to\infty} \widehat{I} = 0$. Thus, by (4.8) and (4.3), we obtain

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K} \left| L(\lambda, \alpha, s + ig(\tau)) - L_n(\lambda, \alpha, s + ig(\tau)) \right| \, \mathrm{d}\tau = 0.$$

Therefore, the definition of the metric d completes the proof of the lemma. □

5 Tightness

Recall that a family of probability measures $\{P\}$ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ is called tight if, for every $\varepsilon > 0$, there exists a compact set $K \subset \mathbb{X}$ such that

$$P(K) > 1 - \varepsilon, \quad \forall P \in \{P\}.$$

Let $P_{n,\lambda,\alpha}$ be the probability measure from Lemma 3.

Lemma 6. *Suppose that $g(\tau) \in U(T_0)$. Then, the sequence of probability measures $\{P_{n,\lambda,\alpha} : n \in \mathbb{N}\}$ is tight.*

Proof. Let K be a compact set of the strip D, and $\mathcal{L}$ is a simple closed curve lying in D and enclosing the set K. Then, in view of the Cauchy integral formula,

$$\sup_{s\in K} \left| L_n(\lambda, \alpha, s + ig(\tau)) \right| \ll \int_{\mathcal{L}} \frac{|L_n(\lambda, \alpha, z + ig(\tau))|}{|s - z|} |\mathrm{d}z|$$
$$\ll \left(\int_{\mathcal{L}} \frac{|\mathrm{d}z|}{|s - z|^2} \int_{\mathcal{L}} \left| L_n(\lambda, \alpha, z + ig(\tau)) \right|^2 |\mathrm{d}z| \right)^{1/2}.$$

Hence,

$$\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K} \left| L_n(\lambda, \alpha, s + ig(\tau)) \right| \mathrm{d}\tau$$
$$\ll_K \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \left(\frac{1}{T} \int_0^T \sup_{s\in K} \left| L_n(\lambda, \alpha, \sigma_K + ig(\tau)) \right|^2 \mathrm{d}\tau \right)^{1/2}, \tag{5.1}$$

where $\sigma_K > 1/2$. Similarly as in Section 4, we find, for $V \geqslant T_0$,

$$\int_V^{2V} \left| L_n(\lambda, \alpha, \sigma_K + ig(\tau)) \right|^2 \mathrm{d}\tau$$
$$\ll \left(\frac{1}{g'(2V)} + \frac{1}{g'(V)} \right) \int_{-g(2V)}^{g(2V)} \left| L_n(\lambda, \alpha, \sigma_K + iu) \right|^2 \mathrm{d}u. \tag{5.2}$$

Since the series for $L_n(\lambda, \alpha, \sigma_K + iu))$ is absolutely convergent, we have

$$\int_{-g(2V)}^{g(2V)} \left| L_n(\lambda, \alpha, \sigma_K + iu) \right|^2 \mathrm{d}u \ll g(2V) \sum_{m=0}^{\infty} \frac{u_n^2(m, \alpha)}{(m + \alpha)^{2\sigma_K}}.$$

Therefore, (5.2) yields

$$\int_V^{2V} \sup_{s\in K} \left|L_n(\lambda,\alpha,\sigma_K+ig(\tau))\right|^2 \,\mathrm{d}\tau \ll \frac{g(2V)}{\min(g'(2V),g'(V))}\sum_{m=0}^{\infty}\frac{u_n^2(m,\alpha)}{(m+\alpha)^{2\sigma_K}}$$
$$\ll V\sum_{m=0}^{\infty}\frac{u_n^2(m,\alpha)}{(m+\alpha)^{2\sigma_K}}.$$

This together with (5.1) show that

$$\sup_{n\in\mathbb{N}}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in K}\left|L_n(\lambda,\alpha,s+ig(\tau))\right| \,\mathrm{d}\tau \ll_K \left(\sum_{m=0}^{\infty}\frac{u_n^2(m,\alpha)}{(m+\alpha)^{2\sigma_K}}\right)^{1/2}$$
$$\ll_K \left(\sum_{m=1}^{\infty}\frac{1}{m^{2\sigma_K}}\right)^{1/2} \leqslant c_K < \infty. \tag{5.3}$$

Introduce a random variable θ_T defined on a certain probability space $(\Xi,\mathcal{A},\nu)$ and uniformly distributed in $[0,T]$. Now, let $K=K_r$, where K_r are compact sets from the definition of the metric d. Fix $\varepsilon>0$ and define $R_r=2^{-r}\varepsilon^{-1}c_{K_r}$. Moreover, define the $\mathcal{H}(D)$-valued random element

$$Y_{T,n,\lambda,\alpha}=Y_{T,n,\lambda,\alpha}(s)=L_n(\lambda,\alpha,s+ig(\theta_T)),$$

and denote by $Y_{n,\lambda,\alpha}=Y_{n,\lambda,\alpha}(s)$ the random element with the distribution $P_{n,\lambda,\alpha}$. Then, the above definitions, (5.3) and Chebyshev's type inequality yield

$$\nu\Big\{\sup_{s\in K_r}|Y_{T,n,\lambda,\alpha}(s)|\geqslant R_r\Big\}=\frac{1}{T}\mathfrak{L}\Big\{\tau\in[0,T]:\sup_{s\in K_r}|L_n(\lambda,\alpha,s+ig(\tau))|\geqslant R_r\Big\}$$
$$\leqslant\frac{1}{TR_r}\int_0^T\sup_{s\in K}|L_n(\lambda,\alpha,s+ig(\tau))|\,\mathrm{d}\tau\leqslant\frac{\varepsilon}{2^r}$$

for all $n\in\mathbb{N}$. Let $\xrightarrow{\mathcal{D}}$ mean the convergence in distribution. Then, Lemma 3 shows that

$$Y_{T,n,\lambda,\alpha}\xrightarrow[T\to\infty]{\mathcal{D}}Y_{n,\lambda,\alpha}. \tag{5.4}$$

Hence,

$$\sup_{s\in K_r}|Y_{T,n,\lambda,\alpha}|\xrightarrow[T\to\infty]{\mathcal{D}}\sup_{s\in K_r}|Y_{n,\lambda,\alpha}|.$$

From this and (5.3), we obtain that

$$\nu\Big\{\sup_{s\in K_r}|Y_{n,\lambda,\alpha}(s)|\geqslant R_r\Big\}\leqslant\frac{\varepsilon}{2^r} \tag{5.5}$$

for all $n\in\mathbb{N}$. Define the set $K=\{g\in\mathcal{H}(D):\sup_{s\in K_r}|g(s)|\leqslant R_r, r\in\mathbb{N}\}$. Then, by compactness principle, the set K is compact in $\mathcal{H}(D)$. Moreover, by (5.5),

$$\nu\{Y_{n,\lambda,\alpha}\in K\}\geqslant1-\varepsilon\sum_{r=1}^{\infty}\frac{1}{2^r}=1-\varepsilon$$

for all $n \in \mathbb{N}$. This and the definition of $Y_{n,\lambda,\alpha}$ prove that $P_{n,\lambda,\alpha}(K) \geqslant 1-\varepsilon$ for all $n \in \mathbb{N}$. The proof is complete. □

6 Limit theorems

In this section, we will prove probabilistic limit theorems for the function $L(\lambda,\alpha,s)$ in the space $\mathcal{H}(D)$. For $A \in \mathcal{B}(\mathcal{H}(D))$, set

$$\widehat{P}_{T,\lambda,\alpha}(A) = \frac{1}{T}\mathfrak{L}\left\{\tau \in [0,T] : L(\lambda,\alpha,s+ig(\tau)) \in A\right\}.$$

Moreover, let

$$L(\lambda,\alpha,\omega,s) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i\lambda m}\omega(m)}{(m+\alpha)^s}.$$

Note that the latter series, for almost all $\omega \in \Omega$, is uniformly convergent on compact subsets of the strip D, thus $L(\lambda,\alpha,\omega,s)$ is a $\mathcal{H}(D)$-valued random element on the space $(\Omega,\mathcal{B}(\Omega),m_H)$ [17]. Denote by P_L the distribution of $L(\lambda,\alpha,\omega,s)$, i.e.,

$$P_L(A) = m_H\left\{\omega \in \Omega : L(\lambda,\alpha,\omega,s) \in A\right\}, \quad A \in \mathcal{B}(\mathcal{H}(D)).$$

Theorem 5. *Suppose that the set $V(\alpha)$ is linearly independent over $\mathbb{Q}$, and $g(\tau) \in U(T_0)$. Then, $\widehat{P}_{T,\lambda,\alpha}$ converges weakly to P_L as $T \to \infty$.*

Proof. Let θ_T, $Y_{T,n,\lambda,\alpha}$ and $Y_{n,\lambda,\alpha}$ be the same as in Section 5. Since, by Lemma 6, the sequence $\{P_{n\,\lambda,\alpha} : n \in \mathbb{N}\}$ is tight, by the Prokhorov theorem [5], it is relatively compact. Therefore, there are a subsequence $\{P_{n_l,\lambda,\alpha}\} \subset \{P_{n,\lambda,\alpha}\}$ and a probability measure $P_{\lambda,\alpha}$ on $(\mathcal{H}(D),\mathcal{B}(\mathcal{H}(D)))$ such that $P_{n_l,\lambda,\alpha}$ converges weakly to $P_{\lambda,\alpha}$ as $l \to \infty$. In other words,

$$Y_{n_l,\lambda,\alpha} \xrightarrow[l\to\infty]{\mathcal{D}} P_{\lambda,\alpha}. \tag{6.1}$$

Introduce one more $\mathcal{H}(D)$-valued random element

$$\widehat{Y}_{T,\lambda,\alpha} = \widehat{Y}_{T,\lambda,\alpha}(s) = L(\lambda,\alpha,s+ig(\theta_T)),$$

fix $\varepsilon > 0$, and apply Lemma 5. This gives

$$\begin{aligned}
&\lim_{l\to\infty}\limsup_{T\to\infty}\nu\left\{d(Y_{T,n,\lambda,\alpha},Y_{n_l,\lambda,\alpha}) \geqslant \varepsilon\right\} \\
&\quad= \lim_{l\to\infty}\limsup_{T\to\infty}\frac{1}{T}\mathfrak{L}\left\{\tau \in [0,T] :\right. \\
&\qquad\qquad\left. d(L(\lambda,\alpha,s+ig(\tau)),L_{n_l}(\lambda,\alpha,s+ig(\tau))) \geqslant \varepsilon\right\} \\
&\quad\leqslant \frac{1}{\varepsilon T}\int_0^T d(L(\lambda,\alpha,s+ig(\tau)),L_{n_l}(\lambda,\alpha,s+ig(\tau)))\,\mathrm{d}\tau = 0.
\end{aligned} \tag{6.2}$$

The space $\mathcal{H}(D)$ is separable and the random elements $Y_{T,n,\lambda,\alpha}$, $Y_{n,\lambda,\alpha}$ and $\widehat{Y}_{T,\lambda,\alpha}$ are defined on the same probability space $(\Xi, \mathcal{A}, \nu)$. Therefore, (5.4), (6.1) and (6.2) show that the above random elements satisfy all hypotheses of Theorem 4.2 of [5]. Thus, we have

$$\widehat{Y}_{T,\lambda,\alpha} \xrightarrow[T\to\infty]{\mathcal{D}} P_{\lambda,\alpha},$$

i.e., $\widehat{P}_{T,\lambda,\alpha}$ converges weakly to $P_{\lambda,\alpha}$ as $T \to \infty$. Moreover, the latter relation shows that the measure $P_{\lambda,\alpha}$ does not depend on the sequence $\{P_{n_l,\lambda,\alpha}\}$. Therefore, relation (6.1) can be replaced by

$$Y_{n,\lambda,\alpha} \xrightarrow[n\to\infty]{\mathcal{D}} P_{\lambda,\alpha},$$

which is equivalent to weak convergence of $P_{n,\lambda,\alpha}$ to $P_{\lambda,\alpha}$ as $n \to \infty$. Thus, the measures $P_{n,\lambda,\alpha}$, as $n \to \infty$, and $\widehat{P}_{T,\lambda,\alpha}$, as $T \to \infty$, have the same limit measure $P_{\lambda,\alpha}$. We notice that, in view of Lemma 4, the measure $P_{n,\lambda,\alpha}$ is the same as in the case $g(\tau) = \tau$.

In [17, Chapter 5], the weak convergence, as $T \to \infty$, for

$$\widetilde{P}_{T,n,\lambda,\alpha}(A) = \frac{1}{T}\mathfrak{L}\left\{\tau \in [0,T] : L(\lambda,\alpha,s+i\tau) \in A\right\}, \quad A \in \mathcal{B}(\mathcal{H}(D)),$$

was considered, and it was obtained that $\widetilde{P}_{T,n,\lambda,\alpha}$ converges weakly to the measure $P_{\lambda,\alpha}$ too, and that $P_{\lambda,\alpha} = P_L$. This proves the theorem. □

Theorem 6. *Suppose that $g(\tau) \in U(T_0)$. Then, on $(\mathcal{H}(D), \mathcal{B}(\mathcal{H}(D)))$ there exists a probability measure $P_{\lambda,\alpha}$ such that $\widehat{P}_{T,\lambda,\alpha}$ converges weakly to $P_{\lambda,\alpha}$ as $T \to \infty$.*

Proof. It coincides with the proof of Theorem 5 without a part devoted to identification of the measure $P_{\lambda,\alpha}$. □

7 Proof of approximation

Theorems 3 and 4 follow easily from Theorems 5 and 6, respectively, and the Mergelyan theorem on approximation of analytic functions by polynomials [26]. We recall the notion of a support of probability measures which also additionally occurs in the proofs of Theorems 3 and 4. Suppose that $\mathbb{X}$ is a separable space, and P is a probability measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. A minimal closed set $S \subset \mathbb{X}$ such that $P(S) = 1$ is called the support of P. The set S consists of all $x \in \mathbb{X}$ such that, for every open neighbourhood G of x, the inequality $P(G) > 0$ is valid.

Proof. *(Proof of Theorem 3)* It is known [17, Chapter 5], that the support of P_L is the set $\mathcal{H}(D)$. Let

$$\mathcal{G}_\varepsilon = \left\{g \in \mathcal{H}(D) : \sup_{s\in K} |g(s) - p(s)| < \varepsilon/2\right\},$$

where $p(s)$ is a polynomial. Since $p(s)$ is an element of the support of P_L, we have

$$P_L(\mathcal{G}_\varepsilon) > 0. \tag{7.1}$$

By the Mergelyan theorem, we choose $p(s)$ satisfying

$$\sup_{s\in K} |f(s) - p(s)| < \varepsilon/2.$$

Then, we have that

$$\mathcal{G}_\varepsilon \subset \left\{ g \in \mathcal{H}(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon \right\} \stackrel{\text{def}}{=} G_\varepsilon.$$

Thus, in view of (7.1),

$$P_L(G_\varepsilon) > 0. \tag{7.2}$$

Since the set G_ε is open, by Theorem 5,

$$\liminf_{T\to\infty} \widehat{P}_{T,\lambda,\alpha}(G_\varepsilon) \geqslant P_L(G_\varepsilon) > 0,$$

and the definitions of $\widehat{P}_{T,\lambda,\alpha}$ and G_ε prove the first statement of the theorem.

For the proof of the second statement of the theorem, we observe that the boundaries of the set G_ε do not intersect for different ε. From this, it follows that the set G_ε is a continuity set of the measure P_L for all but at most countably many $\varepsilon > 0$. Therefore, by Theorem 5 and (7.2), the limit

$$\lim_{T\to\infty} \widehat{P}_{T,\lambda,\alpha}(G_\varepsilon) = P_L(G_\varepsilon)$$

exists and is positive for all but at most countably many $\varepsilon > 0$. □

Proof. *(Proof of Theorem 4)* Let $\mathfrak{F}_{\lambda,\alpha}$ denote the support of the limit measure $P_{\lambda,\alpha}$ in Theorem 6. Clearly $\mathfrak{F}_{\lambda,\alpha}$ is a non-empty closed set. Since $f(s) \in \mathfrak{F}_{\lambda,\alpha}$, an analogue of inequality (7.2) for $P_{\lambda,\alpha}$ is true. Therefore, by Theorem 6,

$$\liminf_{T\to\infty} \widehat{P}_{T,\lambda,\alpha}(G_\varepsilon) \geqslant P_{\lambda,\alpha}(G_\varepsilon) > 0,$$

and we have the first statement of theorem.

Similarly, as in the proof of Theorem 3, the set G_ε is a continuity set of the measure $P_{\lambda,\alpha}$ for all but at most countably many $\varepsilon > 0$. Thus, by Theorem 6, the limit

$$\lim_{T\to\infty} \widehat{P}_{T,\lambda,\alpha}(G_\varepsilon) = P_{\lambda,\alpha}(G_\varepsilon)$$

exists and is positive for all but at most countably many $\varepsilon > 0$. The theorem is proved. □

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