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On Well Posed Impulsive Boundary Value Problems for $p(t)$-Laplacian's

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Abstract. In this paper we investigate via variational methods and critical point theory the existence of solutions, uniqueness and continuous dependence on parameters to impulsive problems with a $p(t)$-Laplacian and Dirichlet boundary value conditions.

Keywords: continuous dependence on parameters, existence, impulsive differential equation, $p(t)$-Laplace operator.

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1 Introduction

Let the numbers $0 < t_1 < t_2 < \cdots < t_m < \pi$ be fixed throughout the paper; $0 < m$ is a fixed integer. Let $I_j : R \to R$ be continuous functions $j = 1, 2, \ldots, m$ and let $f : [0, \pi] \times R \times R \to R$ be a Caratheodory function. Let $p \in C(0, \pi)$, $s \in L^\infty_+(0, \pi)$, where

$$L^\infty_+(0,\pi) = \Big\{ v \in L^\infty(0,\pi) \;\Big|\; \operatorname*{ess\,inf}_{x\in[0,\pi]} v(x) \ge 1 \Big\}.$$

We assume $p^- = \inf_{t\in[0,\pi]} p(t) > 1$, $s^- > 1$. We will consider parameters $u \in L^{s(t)}(0, \pi)$ such that $\|u\|_{L^{s(t)}} \le M$ for some fixed $M > 0$.

In this paper we consider the following impulsive BVP in $X = W_0^{1,p(t)}(0, \pi)$

$$\begin{aligned} &-\frac{d}{dt}\left(\left|\frac{d}{dt}x(t)\right|^{p(t)-2}\frac{d}{dt}x(t)\right) + f\big(t, x(t), u(t)\big) = g(t), \\ &x(0) = x(\pi) = 0 \end{aligned} \tag{1.1}$$

subject also to the impulsive condition

$$\begin{aligned}\left|\frac{d}{dt}x(t_j^+)\right|^{p(t)-2}\frac{d}{dt}x(t_j^+)-\left|\frac{d}{dt}x(t_j^-)\right|^{p(t)-2}\frac{d}{dt}x(t_j^-)\\ = I_j\big(x(t_j)\big) \text{ for } j=1,2,\dots,m,\end{aligned} \tag{1.2}$$

where it is assumed that both limits

$$\lim_{t\to t_j^+}\left|\frac{d}{dt}x(t)\right|^{p(t)-2}\frac{d}{dt}x(t),\quad \lim_{t\to t_j^-}\left|\frac{d}{dt}x(t)\right|^{p(t)-2}\frac{d}{dt}x(t)$$

exist.

Dirichlet problems with a $p(\cdot)$-Laplacian arise naturally in mathematical models in elastic mechanics (see [29]), electrorheological fluids (see [19]) and image restoration (see [6]). It is of interest to know conditions which guarantee

a) existence of solutions,

b) uniqueness,

c) continuous dependence of the solutions on parameters.

Problems satisfying all three conditions are called well-posed. To study well-posed problems with rather mild assumptions we apply a direct variational method. Let us consider the following simple example taken from [10]. The non-impulsive problem

$$-\ddot{x}(t)+\lambda x(t)=f(t),\quad x\in H_0^1(0,\pi)$$

for any $f\in L^1(0,\pi)$ has a unique classical solution with $\lambda>-1$. On the other hand let us consider problem

$$\begin{aligned}-\ddot{x}(t)=0,\qquad x(0)=x(\pi)=0,\\ \dot{x}(1^+)-\dot{x}(1^-)=\frac{1}{3}x^3(1)-4x(1)\end{aligned} \tag{1.3}$$

with one impulse at $t_1=1$. We see that the solution to $-\ddot{x}(t)=0$ on $[0,1)$ satisfying $x(0)=0$ is $x(t)=\alpha t$ and on $(1,\pi]$ the solution is $x(t)=\beta t+\gamma$. Since x is continuous we obtain

$$\beta\pi+\gamma=0\quad\text{and}\quad \alpha=\beta+\gamma$$

taking into account the boundary conditions. This results in $\alpha=-\frac{1}{\pi}(\gamma-\pi\gamma)$, $\beta=-\frac{1}{\pi}\gamma$ and $x(1)=-\frac{1}{\pi}(\gamma-\pi\gamma)$. From the impulsive condition we get

$$-\gamma=\frac{1}{3}\left(\frac{1}{\pi}(\gamma-\pi\gamma)\right)^3-4\left(\frac{1}{\pi}(\gamma-\pi\gamma)\right). \tag{1.4}$$

One can show that equation (1.4) has three solutions in $[-6,6]$ and we see that these solutions are

$$\gamma=0,\quad \gamma=-\sqrt{3}\pi\frac{\sqrt{3\pi-4}}{(\pi-1)^{\frac{3}{2}}},\quad \gamma=\sqrt{3}\pi\frac{\sqrt{3\pi-4}}{(\pi-1)^{\frac{3}{2}}}. \tag{1.5}$$

Summarizing, the solution to (1.3) is a function

$$x(t) = \begin{cases} -\dfrac{1}{\pi}(\gamma - \pi\gamma)t & \text{for } t \in [0,1], \\ -\dfrac{1}{\pi}\gamma t + \gamma & \text{for } t \in (1,\pi] \end{cases}$$

with any γ satisfying (1.5). In this case

$$v \to \int_0^v \left(\frac{1}{3}w^3 - 4w\right) dw = \frac{1}{12}v^4 - 2v^2,$$

which is an antiderivative of an impulsive function and it is not a convex function. It is well known that convexity is related to uniqueness in variational problems. We are interested to study well posed impulsive variational problems when the relevant properties of the non-impulsive problems are retained.

The study of impulsive boundary value problems is important because of its applications to problems in which abrupt changes appear at certain times in the evolution process, see for example [1, 7, 13, 16, 21, 27]. Such problems arise in medicine, ecology and chemistry. While the literature on impulsive differential equations is rather vast, there are only a few papers concerning the variational approach. Periodic solutions with impulses are considered by critical point theory in [28] within the framework sketched in [17]. Moreover, in [18,25] impulsive Duffing type equations with Dirichlet boundary conditions are considered. Multiplicity results are investigated for example in [4] with the aid of Clark's Theorem and in [22] by the fountain theorem. In [11] the variational framework for the Sturm–Liouville boundary value problem is developed for the second order impulsive ordinary differential equation of p-Laplacian type independently from [17]. In [5] existence results for boundary value problem with a p-Laplacian type operator are obtained by using the least action principle and the saddle point theorem, with or without impulsive effects improving some existing results in the literature. Our results deal with more complicated differential operators and thus they are different from those of [5]. In [23] the authors study the existence of infinitely many solutions for a class of second-order impulsive Hamiltonian systems. They obtain some new existence criteria. Using the ideas of Ricceri, in [24] the authors obtain results guaranteeing that the impulsive Hamiltonian systems with a perturbed term have at least three solutions. In [2] the author considers the case when impulses are superlinear. The existence of solutions is reached via mountain pass technique. New types of impulsive problems have been started in [7] where the impulse depends also on a current state of the problem under consideration. In [20] the abstract framework applicable for impulsive problems is sketched in terms of variational inequalities.

In this paper employing direct variational method we consider the existence and uniqueness of solutions to (1.1)–(1.2) and next we investigate what happens as the parameter function u varies. In doing so we consider both the case of strongly and weakly convergent sequence of parameters. Concerning the continuous dependence on parameters we adopt an iterative procedure which is quite different from ideas in [14]. Some ideas which we use come from [10], where these were applied for a second order semilinear impulsive BVP. Since $p(t)$-Laplacian has more complicated nonlinearity we had to use different technical tools in this paper.

2 Mathematical Preliminaries

For some background material concerning Orlicz–Sobolev spaces we refer to [8] and [12,15]. Let $a \in L_+^\infty(0,\pi)$. Following [8] and we define the Lebesgue–Orlicz space

$$L^{a(t)}(0,\pi) = \left\{ x \;\middle|\; x : [0,\pi] \to R \text{ is measurable, } \int_0^\pi |x(t)|^{a(t)}\, dt < +\infty \right\}$$

equipped with a norm

$$\|x\|_{L^{a(t)}} = \inf \left\{ \lambda > 0 \;\middle|\; \int_0^\pi |x(t)/\lambda|^{a(t)}\, dt \le 1 \right\}.$$

We note if $a^- = \operatorname{ess\,inf}_{t\in[0,\pi]} p(t) > 1$ and $a^+ = \operatorname{ess\,sup}_{t\in[0,\pi]} a(t) < +\infty$ then [8] $L^{a(t)}(0,\pi)$ is a reflexive uniformly convex Banach space.

Let $p, q \in C([0,\pi])$, $1/p(t) + 1/q(t) = 1$ for $t \in [0,\pi]$ and we assume $p^- > 1$. Then $W^{1,p(t)}(0,\pi)$ is the generalized Orlicz–Sobolev space, namely

$$W^{1,p(t)}(0,\pi) = \left\{ x \;\middle|\; x \in L^{p(t)}(0,\pi),\ \int_0^\pi \left| \frac{d}{dt} x(t) \right|^{p(t)} dt < +\infty \right\},$$

where the derivative $\frac{d}{dt}$ stands for the weak one – compare with [3] – i.e. $\frac{d}{dt}x$ is an element of $L^{p(t)}(0,\pi)$ which satisfies

$$\int_0^\pi \frac{d}{dt} x(t) y(t)\, dt = -\int_0^\pi x(t) \frac{d}{dt} y(t)\, dt$$

for all $y \in C_0^\infty(0,\pi)$. Any function belonging to $W^{1,p(t)}(0,\pi)$ is in fact absolutely continuous and so the weak derivative is considered as an a.e. derivative. $W^{1,p(t)}(0,\pi)$ has the following norm:

$$\|x\|_{W^{1,p(t)}} = \sqrt{\left\| \frac{dx}{dt} \right\|_{L^{p(t)}} + \|x\|_{L^{p(t)}}}. \tag{2.1}$$

Now $W_0^{1,p(t)}(0,\pi)$ is the closure of $C_0^\infty(0,\pi)$ in $W^{1,p(t)}(0,\pi)$, see [8]. We will let $X = W_0^{1,p(t)}(0,\pi)$. The norm in X is

$$\|x\|_X = \left\| \frac{dx}{dt} \right\|_{L^{p(t)}},$$

which is equivalent to (2.1). Note X is a uniformly convex, reflexive Banach space. Moreover, from [8] we see that there exist constants $C_1^p, C_2^p > 0$ depending on the interval $[0,\pi]$ and on the function p, such that (Poincaré type inequality)

$$\|x\|_{L^{p(t)}} \le C_1 \|x\|_X \quad \text{for all } x \in W_0^{1,p(t)}(0,\pi)$$

and (Sobolev's type inequality)

$$\max_{t\in[0,\pi]} \big|x(t)\big| \le C_2 \left\| \frac{dx}{dt} \right\|_{L^{p(t)}} \qquad \text{for all } x \in W_0^{1,p(t)}(0,\pi). \tag{2.2}$$

The functional $x \to \int_0^\pi \left|\frac{d}{dt}x(t)\right|^{p(t)} dt$ is called a modular for X. Its Gâteaux derivative $L : X \to X^*$ at any $x \in W_0^{1,p(t)}(0,\pi)$ is given by

$$\big\langle L(x), y\big\rangle = \int_0^\pi \left|\frac{dx(t)}{dt}\right|^{p(t)-2} \frac{dx(t)}{dt}\,\frac{dy(t)}{dt}\,dt \quad \text{for all } y \in X.$$

We have the following relation between a modular and a norm

$$\min\big\{\|u\|_X^{p^-}, \|u\|_X^{p^+}\big\} \le \int_0^\pi \left|\frac{du(t)}{dt}\right|^{p(t)} dt \le \max\big\{\|u\|_X^{p^-}, \|u\|_X^{p^+}\big\}. \tag{2.3}$$

In proving that a weak solution is a classical one we shall use the following version of the Fundamental Theorem of the Calculus of Variation.

Lemma 1. **[26]** *If $g, h \in L^1(0,\pi)$ and*

$$\int_0^\pi \left(g(t)y(t) + h(t)\frac{dy(t)}{dt}\right) dt = 0$$

for all $y \in C_0^\infty(0,\pi)$, then $\frac{d}{dt}h = g$ a.e. on $[0,\pi]$ and $\frac{d}{dt}h \in L^1(0,\pi)$.

From Lemma 1 it follows that $h(t) = \int_0^t g(s)\,ds + c$ for some constant c and for a.e. $t \in [0,\pi]$. Thus g is a classical almost everywhere derivative.

We will require also the following version of the generalized Krasnosel'skii Theorem on the continuity of the Niemytskij operator. This result is a special case of Theorem 2.1 from [9].

Theorem 1. *Assume that $f : [0,\pi] \times R \times R \to R$ is a Caratheodory function. Let $p, s, q \in L_+^\infty(0,\pi)$. If for any sequence $\{(x_k, y_k)\}_{k=1}^\infty$ convergent to $(\overline{x}, \overline{y}) \in L^{p(\mathbf{t})}(0,\pi) \times L^{s(\mathbf{t})}(0,\pi)$ there exists a function $h \in L^{q(\mathbf{t})}(0,\pi)$ with*

$$\big|f\big(t, x_k(t), y_k(t)\big)\big| \le h(t), \quad \textit{for } k = 1, 2, \ldots \textit{ and a.e. } t \in [0,\pi],$$

then the Niemytskij operator induced by f

$$N_f : L^{p(\mathbf{t})}(0,\pi) \times L^{s(\mathbf{t})}(0,\pi) \ni (x,y) \longmapsto f\big(\cdot, x(\cdot), y(\cdot)\big) \in L^{q(x)}(0,\pi),$$

is well defined and continuous.

3 The Assumptions and Variational Framework

Put $F(t,x,u) = \int_0^x f(t,\xi,u)\,d\xi$ for *a.e.* $t \in [0,\pi]$, $u \in R$ and assume that $F : [0,\pi] \times R \times R \to R$ is a Caratheodory function. Throughout the paper we assume that

$\mathcal{H}1$ the functions $I_j : R \to R$ for $j = 1, 2, \dots, m$ are continuous and nondecreasing;

$\mathcal{H}2$ for any fixed $u \in R$ and a.e. $t \in [0, \pi]$ the function $x \to F(t, x, u)$ is convex;

$\mathcal{H}3$ for each $r > 0$ there exist functions f_r, $g_r \in L^1(0, \pi)$ such that for all $(x, u) \in X \times L^{s(t)}(0, \pi)$ satisfying $\|x\|_X \leq r$, $\|u\|_{L^{s(t)}} \leq M$ and for *a.e.* $t \in [0, \pi]$ we have

$$\left|F\big(t, x(t), u(t)\big)\right| \leq f_r(t), \qquad \left|f\big(t, x(t), u(t)\big)\right| \leq g_r(t).$$

Function $g \in L^1(0, \pi)$ be fixed and such that $g(t) \neq 0$ for a.e. $t \in [0, \pi]$.

As it is typical for variational problems, we consider weak and classical solutions. A function $x \in X$ is a weak solution to (1.1)–(1.2) if it satisfies

$$\sum_{j=0}^{m} \int_{t_j}^{t_{j+1}} \left|\frac{dx(t)}{dt}\right|^{p(t)-2} \frac{dx(t)}{dt} \frac{dv(t)}{dt}\, dt + \sum_{j=1}^{m} I_j\big(x_j(t_j)\big) v(t_j) + \int_0^{\pi} f\big(t, x(t), u(t)\big) v(t)\, dt = \int_0^{\pi} g(t) v(t)\, dt \tag{3.1}$$

for all $v \in X$.

A function $x \in X$ is called a classical solution to (1.1)–(1.2) if it is a weak solution such that the function $|\frac{d}{dt}x(\cdot)|^{p(\cdot)-2}\frac{d}{dt}x(\cdot)$ is absolutely continuous on (t_j, t_{j+1}) for $j = 0, 1, \dots, m$, the limits in (1.2) are defined, and the relation (1.2) holds together with the boundary condition $x(0) = x(\pi) = 0$ and moreover (1.1) is satisfied for a.e. $t \in [0, \pi] \backslash \{t_1, t_2, \dots, t_m\}$ and

$$\frac{d}{dt}\left(\left|\frac{dx(t)}{dt}\right|^{p(t)-2} \frac{dx(t)}{dt}\right) \in L^1(0, \pi).$$

The action functional $J : X \to R$ corresponding to (1.1)–(1.2) is

$$J(x) = \int_0^{\pi} \frac{1}{p(t)} \left|\frac{dx(t)}{dt}\right|^{p(t)} dt + \sum_{j=1}^{m} \int_0^{x(t_j)} I_j(t)\, dt + \int_0^{\pi} F\big(t, x(t), u(t)\big)\, dt - \int_0^{\pi} g(t) x(t)\, dt. \tag{3.2}$$

The idea of taking (3.2) as the action functional originates from [17] and relies on the fact that one must include the impulsive phenomena into the functional of action. Broadly speaking the impulses must somehow appear in the Gâteaux derivative of a functional, so that we must include them into the functional itself.

4 Auxiliary Results

Lemma 2. *Let $u \in L^{s(t)}(0, \pi)$ be fixed. If $\overline{x} \in X$ is a weak solution to* (1.1)–(1.2)*, then it is also a classical one.*

Proof. The ideas of the proof come from [17] so we give a sketch only. Let a function $\overline{x}$ satisfies (3.1). We take any interval (t_j, t_{j+1}) and a function $h \in C_0^\infty(t_j, t_{j+1})$ extended to $C_0^\infty(0,\pi)$ by taking 0 outside (t_j, t_{j+1}). Then we have in (3.1)

$$\int_{t_j}^{t_{j+1}} \left|\frac{d\overline{x}(t)}{dt}\right|^{p(t)-2} \frac{d\overline{x}(t)}{dt}\frac{dh(t)}{dt}\,dt + \int_{t_j}^{t_{j+1}} \big(f\big(t, \overline{x}(t), u(t)\big) - g(t)\big)h(t)\,dt = 0.$$

Since obviously $X \subset L^1(0,\pi)$ it follows from $\mathcal{H}3$ that

$$f\big(\cdot, \overline{x}(\cdot), u(\cdot)\big)\big|_{(t_j,t_{j+1})} - g(\cdot) \in L^1(t_j, t_{j+1}).$$

By Lemma 1 it follows that $\frac{d}{dt}(|\frac{d}{dt}\overline{x}(t)|^{p(t)-2}\frac{d}{dt}\overline{x}(t))$ exists for a.e. $t \in (t_j, t_{j+1})$ and belongs to $L^1(t_j, t_{j+1})$. Next we obtain

$$\begin{aligned}&\int_0^\pi \left|\frac{d\overline{x}(t)}{dt}\right|^{p(t)-2} \frac{d\overline{x}(t)}{dt}\frac{dh(t)}{dt}\,dt + \int_0^\pi \big(f\big(t, \overline{x}(t), u(t)\big) - g(t)\big)h(t)\,dt\\ &\quad+\sum_{j=1}^m \left(\left|\frac{d}{dt}\overline{x}(t_j^+)\right|^{p(t)-2}\frac{d}{dt}\overline{x}(t_j^+) - \left|\frac{d}{dt}\overline{x}(t_{j+1}^-)\right|^{p(t)-2}\frac{d}{dt}\overline{x}(t_{j+1}^-)\right)h(t_j) = 0.\end{aligned}$$

Since $\overline{x}$ is a weak solution we get from equating the above relation and (3.1) that

$$\begin{aligned}&\sum_{j=1}^m \left(\left|\frac{d}{dt}\overline{x}(t_j^+)\right|^{p(t)-2}\frac{d}{dt}\overline{x}(t_j^+) - \left|\frac{d}{dt}\overline{x}(t_{j+1}^-)\right|^{p(t)-2}\frac{d}{dt}\overline{x}(t_{j+1}^-)\right)h(t_j)\\ &\quad= \sum_{j=1}^m I_j\big(\overline{x}(t_j)\big)h(t_j).\end{aligned}$$

Hence for all $j = 1, 2, \ldots, m$ we have

$$\left|\frac{d}{dt}\overline{x}(t_j^+)\right|^{p(t)-2}\frac{d}{dt}\overline{x}(t_j^+) - \left|\frac{d}{dt}\overline{x}(t_j^-)\right|^{p(t)-2}\frac{d}{dt}\overline{x}(t_j^-) = I_j\big(\overline{x}(t_j)\big). \qquad \square$$

5 Existence and Uniqueness

Lemma 3. *Assume that conditions $\mathcal{H}1$–$\mathcal{H}3$ hold. Let $u \in L^{s(t)}(0,\pi)$ be fixed. Then J is Gâteaux differentiable, weakly l.s.c. and coercive and its critical points correspond to the classical solutions of* (1.1)–(1.2).

Proof. By assumption $\mathcal{H}3$ we see that J is well defined on X. Again by $\mathcal{H}3$ we see that J is Gâteaux differentiable. Let us take an arbitrary $x \in X$ and fix $h \in X$. Then the Gâteaux derivative is

$$\begin{aligned}J'(x;h) &= \int_0^\pi \left|\frac{d}{dt}x(t)\right|^{p(t)-2}\frac{d}{dt}x(t)\frac{d}{dt}h(t)\,dt\\ &\quad+\sum_{j=1}^m I_j\big(x(t_j)\big)h(t_j) + \int_0^\pi f\big(t, x(t), u(t)\big)h(t)\,dt - \int_0^\pi g(t)h(t)\,dt.\end{aligned}$$

Therefore each critical point of J is a weak solution of (1.1)–(1.2).

Let us take any weakly convergent sequence $\{x_k\}_{k=1}^{\infty} \subset X$. Then the sequence $\{x_k\}_{k=1}^{\infty}$ has a subsequence $\{x_{k_n}\}_{n=1}^{\infty}$ which is strongly convergent in $L^{p(t)}(0,\pi)$ and convergent in $C[0,\pi]$. Denote by $\overline{x} \in X$ the weak limit of $\{x_{k_n}\}_{n=1}^{\infty}$. Then by continuity we see that

$$\sum_{j=1}^{m} \int_0^{x_{k_n}(t_j)} I_j(t)\,dt \to \sum_{j=1}^{m} \int_0^{\overline{x}(t_j)} I_j(t)\,dt.$$

There exists $r > 0$ such that $\|x_{k_n}\|_X \le r$ for all $n \in N$. Thus from $\mathcal{H}3$ there exists a function $g_r \in L^1(0,\pi)$ such that

$$\left|F\big(t, x_{k_n}(t), u(t)\big)\right| \le g_r(t) \quad \text{for } a.e.\ t \in [0,\pi].$$

Then by the Lebesgue Dominated Convergence Theorem we get

$$\int_0^{\pi} F\big(t, x_{k_n}(t), u(t)\big)\,dt \to \int_0^{\pi} F\big(t, \overline{x}(t), u(t)\big)\,dt.$$

Therefore, J is weakly l.s.c. on X.

Since F is convex in the second variable we see that for all $y, u \in R$ and a.e. $t \in [0,\pi]$ that

$$F(t,y,u) \ge F_x(t,0,u)y + F(t,0,u).$$

There exist functions f_0, $g_0 \in L^1(0,\pi)$ such that

$$\left|F\big(t,0,u(t)\big)\right| \le f_0(t), \quad \left|F_x\big(t,0,u(t)\big)\right| \le g_0(t).$$

We see from (2.2) that

$$\int_0^{\pi} F_x\big(t,0,x(t)\big)x(t)\,dt \le \|x\|_{\infty} \int_0^{\pi} \left|F_x\big(t,0,x(t)\big)\right| dt \le \left(C_2 \int_0^{\pi} g_0(t)\,dt\right) \|x\|_X$$

for any $x \in X$. Thus

$$\int_0^{\pi} F\big(t,x(t),u(t)\big)\,dt \ge -\left(C_2 \int_0^{\pi} g_0(t)\,dt\right) \|x\|_X - \int_0^{\pi} f_0(t)\,dt$$

for any $x \in X$. Since $x \to \int\limits_0^x I_j(t)\,dt$ is convex for $j = 1, 2, \ldots, m$ we see that

$$\int_0^{x(t_j)} I_j(t)\,dt \ge I_j(0)x(t_j) \ge -C_2 \sum_{j=1}^{m} \left|I_j(0)\right| \|x\|_X\,.$$

Finally, we see from (2.3) that for any $x \in X$ with $\|x\|_X \ge 1$

$$\begin{aligned} J(x) \ge{}& \frac{1}{p^+}\|x\|_X^{p^-} - \left(C_2 \int_0^{\pi} g_0(t)\,dt\right)\|x\|_X - \int_0^{\pi} f_0(t)\,dt \\ &- C_2 \sum_{j=1}^{m} \left|I_j(0)\right| \|x\|_X - C_2 \|x\|_X \|g\|_{L^1}\,. \end{aligned} \tag{5.1}$$

Also since $p^- > 1$ we see that J is coercive. □

Theorem 2. *Assume that $\mathcal{H}1$–$\mathcal{H}3$ hold. Let $u \in L^{s(t)}(0,\pi)$ be fixed. Problem* (1.1)–(1.2) *has exactly one solution $\overline{x}_u \in X$ such that $\overline{x}_u(t) \neq 0$ for a.e. $t \in [0,\pi]$.*

Proof. By Lemma 3 J is Gâteaux differentiable, weakly l.s.c. and coercive on X. Therefore there exists $\overline{x}_u \in X$ such that $J(\overline{x}_u) = \inf_{v\in X} J(v)$ and thus $\overline{x}_u$ satisfies (3.1). Note that the functional

$$J_1(x) = \int_0^\pi \frac{1}{p(t)} \left| \frac{d}{dt} x(t) \right|^{p(t)} dt$$

is strictly convex. Since I_j are nondecreasing, functions it follows that $x \to \int_0^x I_j(t)\,dt$ are convex. Since F is convex in the second variable, therefore $x \to \int_0^\pi F(t,x(t),u(t))\,dt$ is a convex functional.

$$J_2(x) = \sum_{j=1}^m \int_0^{x(t_j)} I_j(t)\,dt + \int_0^\pi F\big(t,x(t),u(t)\big)\,dt - \int_0^\pi g(t)x(t)\,dt$$

is convex, we see that $J = J_1 + J_2$ is strictly convex. Thus the critical point is unique. An application of Lemma 2 shows that $\overline{x}_u$ is a classical solution. Suppose $\overline{x}_u = 0$ for a.e. $t \in [0,\pi]$. Inserting $\overline{x}_u = 0$ into (1.1) provides that $g = 0$ which is a contradiction. □

6 Continuous Dependence on Parameters

Having shown the existence and uniqueness of a solution, we investigate the dependence on a sequence of parameters. We improve earlier results [10] by showing the strong convergence of the sequence of solutions instead of weak convergence. We also generalize the results from [10] to the case of variable exponent. However we shall need some additional assumption which is not required when we want to obtain the weak convergence of the sequence of solutions.

6.1 Strongly convergent sequence of parameters

Now we replace $\mathcal{H}3$ with the following assumption

$\mathcal{H}4$ $g \in L^{\alpha(t)}(0,\pi)$ where $\alpha \in L_+^\infty(0,\pi)$ with $\alpha^- > 1$, $\alpha^+ < +\infty$ and for each $r > 0$ there exist functions $f_r \in L^1(0,\pi)$, $g_r \in L^{\alpha(t)}(0,\pi)$, such that for all $(x,u) \in X \times L^{s(t)}(0,\pi)$ satisfying $\|x\|_X \le r$, $\|u\|_{L^{s(t)}} \le M$ and for *a.e.* $t \in [0,\pi]$ we have

$$\big|F\big(t,x(t),u(t)\big)\big| \le f_r(t), \qquad \big|f\big(t,x(t),u(t)\big)\big| \le g_r(t).$$

Theorem 3. *Assume that conditions $\mathcal{H}1$, $\mathcal{H}2$, $\mathcal{H}4$ hold. Let $\{u_n\}_{n=1}^\infty$ satisfy $u_n \to u_0$ (strongly) in $L^{s(t)}(0,\pi)$. Then, for any sequence $\{x_n\}_{n=1}^\infty$ of (unique) solutions to* (1.1)–(1.2) *corresponding to u_n, there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty \subset X$ and an element $x_0 \in X$ such that $x_{n_k} \to x_0$ (strongly) in X and x_0 is a classical solution to* (1.1)–(1.2) *corresponding to u_0.*

Proof. We shall apply an iterative technique. We define a sequence $\{x_n\}_{n=1}^{\infty}$, where x_n is a solution to (1.1)–(1.2) with $u = u_n$. Thus

$$\begin{aligned}
&-\frac{d}{dt}\left(\left|\frac{d}{dt}x_n(t)\right|^{p(t)-2}\frac{d}{dt}x_n(t)\right) + f\big(t, x_n(t), u_n(t)\big) = g(t),\\
&x_n(0) = x_n(\pi) = 0,\\
&\left|\frac{d}{dt}x_n(t_j^+)\right|^{p(t)-2}\frac{d}{dt}x_n(t_j^+) - \left|\frac{d}{dt}x_n(t_j^-)\right|^{p(t)-2}\frac{d}{dt}x_n(t_j^-)\\
&\qquad = I_j\big(x_n(t_j)\big) \quad \text{for } j = 1, 2, \ldots, m.
\end{aligned} \tag{6.1}$$

We now show there exists a constant $r > 0$ such that $\|x_n\|_X \leq r$ for $n \in N$. To see this we just note if $\|x_n\|_X \geq 1$ then by (5.1) we have

$$\begin{aligned}
J(x_n) \geq \frac{1}{p^+}\|x_n\|_X^{p^-} &- C_2\left(\int_0^\pi g_0(t)\,dt + \|g\|_{L^1}\right)\|x_n\|_X - \int_0^\pi f_0(t)\,dt\\
&- C_2\sum_{j=1}^m \big|I_j(0)\big|\,\|x_n\|_X\,.
\end{aligned}$$

Next we see that

$$\begin{aligned}
0 = J(0) \geq \inf_{x\in X} J(x) = J(x_n) &\geq \frac{1}{p^+}\|x_n\|_X^{p^-}\\
&- C_2\left(\int_0^\pi g_0(t)\,dt + \sum_{j=1}^m \big|I_j(0)\big| + \|g\|_{L^1}\right)\|x\|_X - \int_0^\pi f_0(t)\,dt.
\end{aligned}$$

Hence $\{x_n\}_{n=1}^{\infty}$ has a weakly convergent subsequence in X (and for simplicity we will denote this subsequence also by $\{x_n\}_{n=1}^{\infty}$). Let its limit be x_0. Note $\{x_n\}_{n=1}^{\infty}$ converges strongly in $L^{p(t)}(0,\pi)$ and also in $C(0,\pi)$. Since each x_n for $n \in N$ is a critical point so we see that for any $l \geq k$ we have

$$0 = \big\langle J'(x_k), x_k\big\rangle - \big\langle J'(x_l), x_l\big\rangle. \tag{6.2}$$

Writing (6.2) explicitly we get

$$\begin{aligned}
0 = &\int_0^\pi \left|\frac{d}{dt}x_k(t)\right|^{p(t)} dt - \int_0^\pi \left|\frac{d}{dt}x_l(t)\right|^{p(t)} dt + \sum_{j=1}^m I_j\big(x_k(t_j)\big)x_k(t_j)\\
&- \sum_{j=1}^m I_j\big(x_l(t_j)\big)x_l(t_j) + \int_0^\pi \Big(f\big(t, x_k(t), u_k(t)\big) - g(t)\Big)x_k(t)\,dt\\
&- \left(\int_0^\pi f\big(t, x_l(t), u_l(t)\big) - g(t)\right)x_l(t)\,dt.
\end{aligned}$$

Since $\|x_n\|_X \leq r$ by assumption $\mathcal{H}4$ there exists a function $h_r \in L^{\alpha(t)}(0,\pi)$ such that

$$\big|f\big(t, x_l(t), u_l(t)\big) - g(t)\big| \leq h_r(t).$$

Thus the sequence $\{f(\cdot, x_n(\cdot), u_n(\cdot)) - g(\cdot)\}_{n=1}^{\infty}$ is bounded in $L^{\alpha(t)}(0,\pi)$ and as a result it has a weakly convergent subsequence

$$\left\{f\left(\cdot, x_{n_k}(\cdot), u_{n_k}(\cdot)\right) - g(\cdot)\right\}_{k=1}^{\infty},$$

whose limit we denote by $h \in L^{\alpha(t)}(0,\pi)$.

Recalling that $\{x_{n_k}\}_{k=1}^{\infty}$ converges strongly in $L^{p(t)}(0,\pi)$ we see that by letting $n_k \to \infty$

$$\int_0^{\pi} \left(f\left(t, x_{n_k}(t), u_{n_k}(t)\right) - g(t)\right) x_{n_k}(t)\, dt \to \int_0^{\pi} h(t) x_0(t)\, dt.$$

Hence letting $l_k, n_k \to \infty$ we see that

$$\int_0^{\pi} \left(f\left(t, x_{n_k}(t), u_{n_k}(t)\right) - g(t)\right) x_{n_k}(t)\, dt \\ - \int_0^{\pi} \left(f\left(t, x_{l_k}(t), u_{l_k}(t)\right) - g(t)\right) x_{l_k}(t)\, dt \to 0.$$

Thus for a fixed $\varepsilon > 0$, there is some N_ε such that for all $l_k \geq n_k \geq N_\varepsilon$

$$-\frac{\varepsilon}{2} \leq \int_0^{\pi} \left(f\left(t, x_{n_k}(t), u_{n_k}(t)\right) - g(t)\right) x_{n_k}(t)\, dt \\ - \int_0^{\pi} \left(f\left(t, x_{l_k}(t), u_{l_k}(t)\right) - g(t)\right) x_{l_k}(t)\, dt \leq \frac{\varepsilon}{2}.$$

Since $\{x_{n_k}\}_{k=1}^{\infty}$ converges strongly in $C(0,\pi)$ we observe that for the same $\varepsilon > 0$, there is some M_ε such that for all $l_k \geq n_k \geq M_\varepsilon$

$$-\frac{\varepsilon}{2} \leq \sum_{j=1}^{m} I_j\left(x_{n_k}(t_j)\right) x_{n_k}(t_j) - \sum_{j=1}^{m} I_j\left(x_{l_k}(t_j)\right) x_{l_k}(t_j) \leq \frac{\varepsilon}{2}.$$

Hence, taking $K_\varepsilon \geq \max\{M_\varepsilon, N_\varepsilon\}$ we see that for all $l_k \geq n_k \geq K_\varepsilon$

$$\left| \int_0^{\pi} \left| \frac{d}{dt} x_{n_k}(t) \right|^{p(t)} dt - \int_0^{\pi} \left| \frac{d}{dt} x_{l_k}(t) \right|^{p(t)} dt \right| \leq \varepsilon. \tag{6.3}$$

Recall that X is a Banach space and that norm convergence is equivalent to the modular convergence, see [8]. From (6.3) and from $x_{n_k} \rightharpoonup x_0$ we have

$$\|x_{n_k}\|_X \to \|x_0\|_X\,.$$

Since X is uniformly convex, we see that $\{x_{n_k}\}_{k=1}^{\infty}$ is strongly convergent to x_0. Note the generalized Krasnosel'skij Theorem (Theorem 1) applies and

$$f\left(\cdot, x_{n_k}(\cdot), u_{n_k}(\cdot)\right) - g(\cdot) \to f\left(\cdot, x_0(\cdot), u_0(\cdot)\right) - g(\cdot)$$

in $L^{\alpha(t)}(0,\pi)$. Since a pair (x_{n_k}, u_{n_k}) satisfies (3.1) we see by taking limits as $n_k \to \infty$ that (x_0, u_0) also satisfies (3.1). By Lemma 2, we see that now x_0 is a classical solution (1.1)–(1.2) corresponding to u_0. $\square$

In the case of a weakly convergent sequence of solutions we do not need the assumption $\mathcal{H}4$.

Theorem 4. *Assume that conditions $\mathcal{H}1$–$\mathcal{H}3$ hold. Let $\{u_n\}_{n=1}^{\infty}$ satisfy $u_n \to u_0$ (strongly) in $L^{s(t)}(0,\pi)$. Then, for any sequence $\{x_n\}_{n=1}^{\infty}$ of solutions to* (1.1)–(1.2) *corresponding to u_n, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset X$ and an element $x_0 \in X$ such that $x_{n_k} \rightharpoonup x_0$ (weakly) in X and x_0 is a classical solution to* (1.1)–(1.2) *corresponding to u_0.*

Proof. We shall again apply an iterative technique. We define a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ by (6.1). As in the proof of Theorem 3 we can show that $\{x_n\}_{n=1}^{\infty}$ has a weakly convergent subsequence in X (and for simplicity we will denote this subsequence also by $\{x_n\}_{n=1}^{\infty}$). Let its limit be x_0. By assumption $\mathcal{H}3$ there exists a function $g_r \in L^1(0,\pi)$ such that

$$\left|f\big(t, x_n(t), u_n(t)\big) - g(t)\right| \le g_r(t).$$

Thus the sequence

$$\left\{f\big(\cdot, x_n(\cdot), u_n(\cdot)\big) - g(\cdot)\right\}_{n=1}^{\infty}$$

is bounded in $L^1(0,\pi)$. Since the sequence $\{(x_n, u_n)\}_{n=1}^{\infty}$ is convergent strongly in $L^{p(t)}(0,\pi) \times L^{s(t)}(0,\pi)$ the generalized Krasnosel'skij Theorem (see Theorem 1) applies and

$$f\big(\cdot, x_n(\cdot), u_n(\cdot)\big) \to f\big(\cdot, x_0(\cdot), u_0(\cdot)\big)$$

in $L^1(0,\pi)$. Thus $\{f(\cdot, x_{n_k}(\cdot), u_{n_k}(\cdot))\}_{k=1}^{\infty}$ has a subsequence convergent a.e. on $[0,\pi]$ to $f(\cdot, x_0(\cdot), u_0(\cdot))$. We multiply the first equation in (6.1) by a test function $C_0^{\infty}(0,\pi)$ and integrate. As a result, following the lines of the proof of Theorem 2 we see that x_0 is a classical solution to (1.1)–(1.2). $\square$

6.2 Weakly convergent sequence of parameters

In Theorems 3 and 4 the convergence of a sequence of parameters was strong convergence. We are now interested in the case when this convergence is weak. However this would require some structure condition on the nonlinear term, i.e.

$$f(t,x,u) = f_1(t,x) + x f_2(t) u.$$

Let $s^*(t)$ be the conjugate exponent to $s(t)$, i.e. $\frac{1}{s^*(t)} + \frac{1}{s(t)} = 1$ for a.e. $t \in [0,\pi]$. Now we replace $\mathcal{H}3$ with the following assumption

$\mathcal{H}5$ $f_1 : [0,\pi] \times R \to R$ is a Caratheodory function, $f_2 \in L^{s^*(t)}(0,\pi)$ and for each $r > 0$ there exist functions f_r, $g_r \in L^1(0,\pi)$ such that

$$\left|F_1\big(t, x(t)\big)\right| \le f_r(t), \qquad \left|f_1\big(t, x(t)\big)\right| \le g_r(t)$$

for all $x \in X$ satisfying $\|x\|_X \le r$, all u with $\|u\|_{L^{s(t)}} \le M$ and for *a.e.* $t \in [0,\pi]$.

Firstly, we follow the ideas of Theorem 4.

Theorem 5. *Assume that conditions $\mathcal{H}1$, $\mathcal{H}2$, $\mathcal{H}5$ hold. Let $\{u_n\}_{n=1}^{\infty}$ satisfy $u_n \rightharpoonup u_0$ (weakly) in $L^{s(t)}(0,\pi)$. Then, for any sequence $\{x_n\}_{n=1}^{\infty}$ of solutions to* (1.1)–(1.2) *corresponding to u_n, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset X$ and an element $x_0 \in X$ such that $x_{n_k} \rightharpoonup x_0$ (weakly) in X and x_0 is a classical solution to* (1.1)–(1.2) *corresponding to u_0.*

Proof. The proof follows similar lines as in the proof of Theorem 4. In fact we get the weak convergence of a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of solutions corresponding to a sequence of parameters. This sequence can be assumed to be convergent strongly in $L^{p(t)}(0,\pi)$.

Since $\{x_{n_k}\}_{k=1}^{\infty}$ is convergent strongly in $L^{p(t)}(0,\pi)$ the generalized Krasnosel'skij Theorem applies and $f_1(\cdot, x_n(\cdot)) \to f_1(\cdot, x_0(\cdot))$ in $L^1(0,\pi)$. Next, since $\{x_{n_k}\}_{k=1}^{\infty}$ is convergent strongly in $L^{p(t)}(0,\pi)$ and $\{u_{n_k}\}_{k=1}^{\infty}$ is convergent weakly in $L^{s(t)}(0,\pi)$ we get

$$\int_0^{\pi} f_2(t)u_{n_k}(t)x_{n_k}(t)\,dt \to \int_0^{\pi} f_2(t)u_0(t)x_0(t)\,dt. \qquad \square$$

The application of the ideas of Theorem 3 requires some special type assumptions on the nonlinear term.

$\mathcal{H}6$ $s^- > 1$, $s^+ < +\infty$, $g \in L^{s(t)}(0,\pi)$ and $f_1 : [0,\pi] \times R \to R$ is a Caratheodory function, $f_2 \in L^{\infty}(0,\pi)$ and for each $r > 0$ there exist functions $f_r \in L^1(0,\pi)$, $g_r \in L^{s(t)}(0,\pi)$ such that

$$\left|F_1(t, x(t))\right| \le f_r(t), \qquad \left|f_1(t, x(t))\right| \le g_r(t)$$

for all $x \in X$ satisfying $\|x\|_X \le r$, $\|u\|_{L^{s(t)}} \le M$ and for *a.e.* $t \in [0,\pi]$.

We have the following result which follows as in the lines of the proof of Theorem 3.

Theorem 6. *Assume that conditions $\mathcal{H}1$, $\mathcal{H}2$, $\mathcal{H}6$ hold. Let $\{u_n\}_{n=1}^{\infty}$ satisfy $u_n \rightharpoonup u_0$ (weakly) in $L^{s(t)}(0,\pi)$. Then, for any sequence $\{x_n\}_{n=1}^{\infty}$ of solutions to (1.1)–(1.2) corresponding to u_n, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset X$ and an element $x_0 \in X$ such that $x_{n_k} \rightharpoonup x_0$ (weakly) in X and x_0 is a classical solution to (1.1)–(1.2) corresponding to u_0.*

Remark 1. We observe that when $s^+ < +\infty$, then in Theorems 5 and 6 we can take a bounded sequence of parameters instead of a weakly convergent one.

7 Examples

We will give some examples of nonlinear problems which can be considered by our methods.

Example 1. Let $g \in L^1(0,\pi)$, $g \neq 0$, $h \in L^{s^*(t)}(0,\pi)$ and let $f : R \to R$ be a nondecreasing function, $f(0) = 0$. Consider

$$-\frac{d}{dt}\left(\left|\frac{d}{dt}x(t)\right|^{p(t)-2}\frac{d}{dt}x(t)\right) + f\big(x(t)\big) + x(t)h(t)u(t) = g(t),$$
$$x(0) = x(\pi) = 0, \quad \dot{x}(1^+) - \dot{x}(1^-) = \frac{1}{3}x^3(1) + 4x(1) \tag{7.1}$$

with one impulse at $t_1 = 1$. We can easily show that conditions $\mathcal{H}1$, $\mathcal{H}2$, $\mathcal{H}5$ are satisfied with $I(v) = \frac{1}{3}v^3 + 4v$. Then the assertion of Theorem 5 holds for problem (7.1). Function 0 cannot be a solution to the above problem since $g \neq 0$.

Example 2. Let $g \in L^1(0,\pi)$, $g \neq 0$, $h \in L^{s^*(t)}(0,\pi)$. Let m be an odd number. Consider

$$-\frac{d}{dt}\left(\left|\frac{d}{dt}x(t)\right|^{p(t)-2}\frac{d}{dt}x(t)\right) + x^m(t)e^{-u^2(t)} + x(t)h(t)u(t) = g(t),$$
$$x(0) = x(\pi) = 0, \quad \dot{x}(1^+) - \dot{x}(1^-) = \frac{1}{3}x^3(1) + 4x(1)$$

with one impulse at $t_1 = 1$. We can easily demonstrate that for the above problem the assertion of Theorem 3 holds. Function 0 cannot be a solution to the above problem since $g \neq 0$.

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