Results for Sixth Order Positively Homogeneous Equations*

T. Garbuza

Daugavpils University

Parades str., Daugavpils, Latvia, LV-5400

E-mail: `garbuza@inbox.lv`

Abstract. We consider positively homogeneous the sixth order differential equations of the type $x^{(6)} = h(t, x)$, where h possesses the property that $h(t, cx) = ch(t, x)$ for $c \geq 0$. This class includes the linear equations $x^{(6)} = p(t)x$ and piece-wise linear ones $x^{(6)} = k_2\, x^+ - k_1\, x^-$. We consider conjugate points and angles associated with extremal solutions and prove some comparison results.

Key words: differential equations of 6-th order, equations with asymmetric nonlinearities, conjugate points, positive homogeneous equations.

1 Introduction

We study sixth-order ordinary differential equations of the type

$$x^{(6)} = h(t, x) \tag{1.1}$$

on a half-axis $[0, +\infty)$, where $h(t, x)$ is a positively homogeneous function, that is $h(t, cx) = ch(t, x)$ for any (t, x) and c nonnegative. Obviously $h(t, 0) \equiv 0$. The linear equation belongs to this class as well as equations of the form $x^{(6)} = k_2 x^+ - k_1 x^-$. The theory of equations of the kind $x^{(n)} = k_2 x^+ - k_1 x^-$ is rich and starts from the works by Fučík and Kufner [7]. For $n = 4$ mention the works by Pope [9] and Krejči [6]. The case of $n = 3$ in a broader setting was studied by Habets and Gaudenzi [3] and Sergejeva [10]. The case $n = 2$ is the most intensively studied. Several generalizations of the equation $x^{(n)} = k_2 x^+ - k_1 x^-$ were considered in a recent paper by Gritsans and Sadyrbaev ([1]).

Respectively the linear theory provides a lot of relevant results. We are interested mostly in oscillatory linear equations. Oscillation of solutions of linear equations is measured in terms of conjugate points (see [11]).

First, we give the definition of a conjugate point (see, Kiguradze [5]).

DEFINITION 1. Let us consider the differential equation

$$x^{(n)} = p(t)x, \quad n > 2, \tag{1.2}$$

which have a zero value at $t = a$. By m-th conjugate point of a with respect to equation (1.2) we call the minimal value of a_{m+n-1}, where a_{m+n-1} is a $(m + n - 1)^{st}$ zero in $[a; +\infty)$ (counting multiplicities) of solutions.

Since conjugate points are infimums of some zeros, and therefore possess some extremal property, the related solutions are called *extremal functions.*

In what follows we shall make this definition more precise for specific equations. The idea of investigation of oscillatory properties of such equations in terms of conjugate points goes back to W. Leighton and Z. Nehari [8], who investigated linear equations of the type $x^{(4)} = p(t)x$, where $p(t)$ is a continuous sign definite function. They formulated a number of results on oscillation of such equations and introduced conjugate points as minima of certain zeros of solutions. They discovered that for $p > 0$ and $p < 0$ the characters of conjugate points essentially differ.

A big part of this theory is valid for equations of the type $x^{(4)} = h(t, x)$, such analysis is done in [4], where one could find basics. It appears that the same results are true for 6-th order positively homogeneous equations.

The theory of conjugate points was developed for equations with increasing right sides, we suppose that p is positive.

The paper is organized as follows. In the second section we discuss the two-termed linear equation $x^{(6)} = k^6 x$, and define conjugate points. The structure of related solutions is studied and some comparison results are provided.

In the third section we consider the Fučík type equation $u^{(6)} = k_2 u^+ - k_1 u^-$ and dual equation $v^{(6)} = k_1 v^+ - k_2 v^-$. We define conjugate points for both equations and study the relation between them (conjugate points).

The fourth section is devoted to positively homogeneous equation. Comparison results are given (Lemma 1 and Lemma 2), which are motivated by similar results in [8].

Theorem 4 gives comparison result for the first extremal solutions of two positively homogeneous equations.

2   Linear Equation

Let us recall ([2, 5]) that conjugate points η_i associated with the point $t = 0$ of the linear equation

$$x^{(6)} = p(t)x$$

form an ascending sequence and coincide with double zeros of solutions x_i which have quadruple zero at $t = 0$. These solutions x_i are uniquely defined (up to multiplication by a constant) by the number $(i - 1)$ of internal (with respect to the interval $(0, \eta_i)$) zeros as well as by the angle

$$\phi_i(k) = \arctan \left(\frac{x_i^{(5)}(0)}{x_i^{(4)}(0)} \right).$$

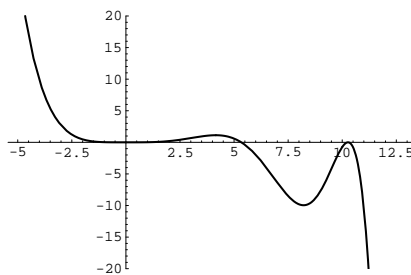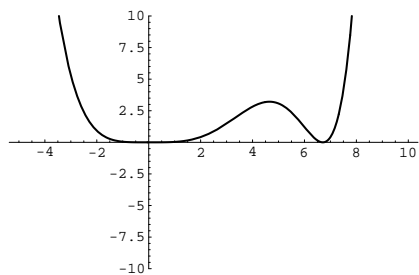They are called extremal solutions. Examples of the first and second solutions are given in Fig. 1, 2.

Figure 1. The first extremal function $x_1(t)$ of (2.1) which relates to η_1, $k = 1$.

Figure 2. The second extremal function $x_2(t)$ of (2.1) which relates to η_2, $k = 1$.

It is known ([2]) that both sequences $\{\eta_i\}$ and $\{\phi_i\}$ are ordered as

$$-\frac{\pi}{2} < \phi_2 < \ldots < \phi_{2n} < \cdots < \phi_{2n+1} < \ldots < \phi_1 < 0, \quad 0 < \eta_1 < \eta_2 < \ldots.$$

Evidently both sequences ϕ_{2n} and ϕ_{2n+1} have limits ϕ_{even} and ϕ_{odd} respectively.

In this section we provide some simple results on the linear equation

$$x^{(6)} = k^6 x, \tag{2.1}$$

where $k \neq 0$. Let us denote by $\eta_i(k)$ conjugate points to the point $t = 0$ of (2.1).

Theorem 1. *If $k_2 > k_1$ then $\eta_i(k_2) < \eta_i(k_1)$ for any i.*

Proof. This result for the 4-th order linear equation was proved in [8] by using the variational method and the Courant minimax eigenvalue theory. We give straightforward proof based on the variable change. Consider the linear equations

$$x^{(6)} = k_2^6 x, \tag{2.2}$$

and

$$x^{(6)} = k_1^6 x, \tag{2.3}$$

where $k_2 > k_1$. By the variable change $\tau = (\frac{k_1}{k_2})t$ the equation (2.3) becomes

$$\frac{d^6 x}{d\tau^6} = \left(\frac{k_2}{k_1}\right)^6 \frac{d^6 x}{dt^6} = k_2^6 x(t) = k_2^6\, x\left(\frac{k_2}{k_1}\tau\right),$$

or

$$\frac{d^6 X}{d\tau^6} = k_2^6 X(\tau), \quad X(\tau) = x\left(\frac{k_2}{k_1}\tau\right)$$

and hence $\eta_i(k_2) = \frac{k_1}{k_2}\eta_i(k_1)$. $\square$

Theorem 2. *If we consider two linear equations* (2.2) *and* (2.3)*, then*

$$\tan \phi_i(k_2) = \left(\frac{k_2}{k_1}\right) \tan \phi_i(k_1). \tag{2.4}$$

Proof. The proof is obtained by computing derivatives of extremal solutions. We use the variable change $t = \frac{k_2}{k_1}\tau$ for the left hand side of (2.4) and have

$$\tan \phi_i(k_2) = \tan \left[\arctan \left.\frac{x_i^{(5)}(\tau)}{x_i^{(4)}(\tau)}\right]\right|_{\tau=0} = \left.\frac{dx_i^{(4)}(\tau)}{d\tau}\frac{1}{x_i^{(4)}(\tau)}\right|_{\tau=0}$$

$$= \left.\frac{dx^{(4)}(t)}{dt}\frac{dt}{d\tau}\frac{1}{x_i^{(4)}(t)}\right|_{\tau=0}^{t=\frac{k_2}{k_1}\tau} = \frac{dx^{(4)}(0)}{dt}\frac{k_2}{k_1}\frac{1}{x_i^{(4)}(0)}$$

$$= \frac{k_2}{k_1}\frac{x_i^{(5)}(0)}{x_i^{(4)}(0)} = \frac{k_2}{k_1}\tan \phi_i(k_1).$$

□

Corollary 1. If we consider two linear equations (2.2) and (2.3) and suppose that $k_2 > k_1$, then $\phi_i(k_2) < \phi_i(k_1)$. The proof follows from Theorem 2 and the fact that $-\pi/2 < \phi_i(k) < 0$ for any i, k.

3   Comparison of Asymmetric Equations

Consider the equation

$$u^{(6)} = k_2\, u^+ - k_1\, u^- \tag{3.1}$$

along with

$$v^{(6)} = k_1\, v^+ - k_2\, v^-, \tag{3.2}$$

where k_1 and k_2 are positive constants. The theory of conjugate points for positively homogeneous equations is applicable for both equations (3.1) and (3.2) as well as for equations (2.2) and (2.3).

Let us use definitions of extremal functions and conjugate points for equations with asymmetric nonlinearities in the right hand side similarly to the case of 4−th order equation (see Henrard and Sadyrbaev [4]).

DEFINITION 2. A solution $x(t)$ of equation (3.1) is called an $i-$**th (+)-extremal** (resp. $i-$**th (−)-extremal**) **function** and denote x_i^+ (resp. x_i^-), if for some $\eta > 0$,

$$x(0) = x'(0) = x''(0) = x'''(0) = x(\eta) = x'(\eta) = 0,$$

$x^{(4)}(0) > 0$, (resp. $x^{(4)}(0) < 0$) and $x(t)$ has exactly $(i-1)$ simple zeros in $(0, \eta)$.

DEFINITION 3. A point η_i^+ (resp. η_i^-) is called an $i-$**th (+)-conjugate** (resp. $i-$**th (−)-conjugate**) **point** (to $t = 0$) with respect to equation (3.1), if there exists a $i-$th (+)-extremal (resp. $i-$th (−)-extremal) function, having quadruple zero at $t = 0$ and double zero at $t = \eta_i^+$ (resp. $t = \eta_i^-$) and exactly $(i-1)$ simple zeros in $(0, \eta_i^+)$ (resp. $(0, \eta_i^-)$).

Remark 1. The (+)-extremal and (−)-extremal functions are defined uniquely by the number of simple zeros up to a positive multiplicative constant.

Let $u_i^{\pm}$ and $v_i^{\pm}$ stand for extremal functions of (3.1) and (3.2) respectively, and $\eta_i^{\pm}$ and $\xi_i^{\pm}$ be associated conjugate points.

Theorem 3. *For equations* (3.1) *and* (3.2) *the following equalities hold:*

$$1)\eta_{2i+1}^+ = \xi_{2i+1}^-, \quad 2)\eta_{2i+1}^- = \xi_{2i+1}^+, \quad 3)\eta_{2i}^- = \xi_{2i}^+, \quad 4)\eta_{2i}^+ = \xi_{2i}^-.$$

Proof. Assertions 1) and 2) follow from the fact that $u_{2i+1}^+ = -v_{2i+1}^-$ and $u_{2i+1}^- = -v_{2i+1}^+$ (by the variable change $u = -v$ equation (3.1) turns to (3.2)). Assertions 3) and 4) follow from the relations $u_{2i}^+(t) = -v_{2i}^-(t)$ and $u_{2i}^-(t) = -v_{2i}^+(t)$. □

Corollary 2. For extremal angles $\phi_i^{\pm}$ and $\psi_i^{\pm}$ associated with equations (3.1) and (3.2) the following equalities are valid:

$$\tan\phi_i^+ = \tan\psi_i^-, \quad \tan\phi_i^- = \tan\psi_i^+.$$

Proof. The proof follows immediately from the observation that $u_i^+(t) = -v_i^-(t)$ and $u_i^-(t) = -v_i^+(t)$. □

Example 1. As illustration we consider two equations

$$u^{(6)} = u^+ - 6u^-, \quad v^{(6)} = 6v^+ - v^-. \tag{3.3}$$

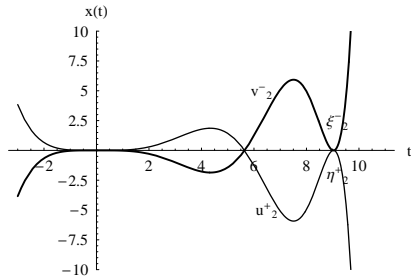In Fig. 4 and Fig. 3 we can see that $\xi_i^- = \eta_i^+$, $i = 1, 2, 3, 4$.

Figure 3. The extremal functions $u_2^+(t)$ of (3.3) which relate to η_2^+ and $v_2^-(t)$ of (??) which relate to ξ_2^-.
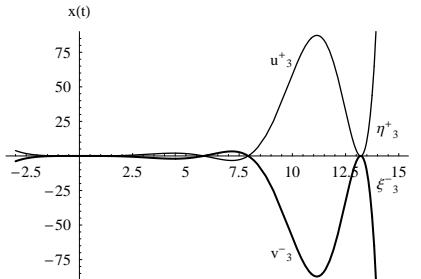
Figure 4. The extremal functions $u_3^+(t)$ of (3.3) which relate to η_3^+ and $v_3^-(t)$ of (??) which relate to ξ_3^-.

4 Comparison of Positively Homogeneous Equations

Consider now equations (1.1) and

$$x^{(6)} = g(t, x), \tag{4.1}$$

where right sides are positively homogeneous continuous functions, strictly increasing in x.

Lemma 1. *Let the inequality $g(t,x) \geq h(t,x)$ holds for any (t,x). Suppose that $x(t)$ and $y(t)$ are solutions of equations (4.1) and (1.1) respectively and*

$$x^{(i)}(0) \geq y^{(i)}(0), \quad i = 0,1,2,3,4,5$$

with at least one inequality being strict. Then $x^{(i)}(t) > y^{(i)}(t)$ for any $t > 0$ and $i = 0,1,2,3,4,5$.

Proof. Consider only the case $x^{(5)}(0) > y^{(5)}(0)$, since the other cases can be treated analogously. Then $x^{(4)}(t) > y^{(4)}(t)$ in some right vicinity of $t = 0$. Suppose $x^{(4)}(t_1) = y^{(4)}(t_1)$ for some $t_1 > 0$ and $x^{(4)}(t) > y^{(4)}(t)$ for any $t \in (0; t_1)$. Then $x^{(i)}(t) > y^{(i)}(t)$, $i = 1,2,3$ and $x(t) > y(t)$ for $t \in (0, t_1]$. We have that in the interval $(0, t_1]$

$$x^{(6)} - y^{(6)} = g(t, x(t)) - h(t, y(t)) > g(t, y(t)) - h(t, y(t)) \geq 0$$

and therefore

$$x^{(i)}(t) > y^{(i)}(t), \quad t \in (0, t_1], \quad i = 5,4,3,2,1,0.$$

We got a contradiction with assumption $x^{(4)}(t_1) = y^{(4)}(t_1)$. Obviously that $x(t) - y(t) > 0$ for $t > 0$. The proof is complete. $\square$

The dual of Lemma 1 states:

Lemma 2. *Let $g(t,x) \geq h(t,x)$. Suppose $x(t)$ and $y(t)$ are solutions of equations (4.1) and (1.1) respectively and*

$$x^{(i)}(b) \geq y^{(i)}(b), \ i = 0,2,4; \quad x^{(j)}(b) \leq y^{(j)}(b), \ j = 1,3,5$$

with at least one inequality being strict. Then for $t < b$ we obtain that

$$x^{(i)}(t) > y^{(i)}(t), \ i = 0,2,4; \quad x^{(j)}(t) < y^{(j)}(t), \ j = 1,3,5.$$

Theorem 4. *Let $g(t,x)$ and $h(t,x)$ be as in Lemma 1. Let $\phi_1^+(g)$ and $\phi_1^+(h)$ be angles in $(-\frac{\pi}{2}, 0)$ (resp.: $\phi_1^-(g)$ and $\phi_1^-(h)$ be angles in $(\frac{\pi}{2}, \pi)$), corresponding to $(+)$-extremal (resp.: $(-)$-extremal) solutions $x_1^+(t)$ and $y_1^+(t)$ (resp.: $x_1^-(t)$ and $y_1^-(t)$). Then $\phi_1^+(g) \leq \phi_1^+(h)$ (resp.: $\phi_1^-(g) \leq \phi_1^-(h)$).*

Proof. Consider the case of extremal functions $x_1(t)$ and $y_1(t)$ associated with differential equations $x^{(6)} = g(t,x)$ and $y^{(6)} = h(t,y)$, respectively. Suppose that $\phi_1^+(g) > \phi_1^+(h)$. Then, by Lemma 1, $x_1^{(j)}(t) > y_1^{(j)}(t)$, $j = 0,1,2,3,4,5$.

For simplicity of notations let us omit lower indices of $x(t)$ and $y(t)$. Denote by $\eta(x)$ and $\eta(y)$ the last double zeros (conjugate points) of $x(t)$ and $y(t)$. Obviously $\eta(x) < \eta(y)$, otherwise $x(t)$ is not greater than $y(t)$ for $t > a$. We got a contradiction, which completes the proof. $\square$

Corollary 3. Let $\phi_1^\pm(k_1)$, $\phi_1^\pm(k_2)$, $\phi_i^\pm(k_{12})$ and $\phi_1^\pm(k_{21})$ stand for the angles associated with first extremal solutions of the equations (2.3), (2.2), (3.2) and (3.1) respectively. Then

$$\frac{\pi}{2} < \phi_1^-(k_{21}) \leq \phi_1^-(k_2) < \phi_1^-(k_1) \leq \phi_1^-(k_{12}) < \pi,$$

$$-\frac{\pi}{2} < \phi_1^+(k_{21}) \leq \phi_1^+(k_2) < \phi_1^+(k_1) \leq \phi_1^+(k_{12}) < 0.$$

Proof.   The proof is obtained by combining Theorem 4 and formula (2.4).   □

Remark 2. We see that the structure of the set of solutions to positively homogeneous equations of order six is generally the same as for the fourth order equations. There are "conjugate points" which measure the rate of oscillation of an equation. These points may be different for solutions which are first positive (in a right neighborhood of $t = a$) and which are first negative. The extremal solutions (those which relate to the conjugate points) behave like extremal solutions for the fourth order equations. They are arranged in a sequence as well as the angles ϕ_i of the initial data. These extremal solutions may be compared. As to comparison of the angles ϕ_i we did it only for the first ones. Description of the rest is still open problem.

The results of this kind may be used for investigation of essentially nonlinear problems. Some steps in this direction were done in the work by the author ([2]).

5   Conclusions

Investigations of equations with asymmetric nonlinearities in the right hand side are useful for many reasons. This type equations have some features of linear ones and at the same time they are nonlinear. They can be investigated by methods similar to those used in the linear theory and specific oscillatory behavior of solutions can be analyzed. Connections between linear and nonlinear theories can be traced. If we wish to compare results for 4th and 6th order positively homogeneous equations, we can see a similar arrangement of angles associated with extremal solutions. It is possible that similar results and some generalizations can be made for the case of even order positively homogeneous equations.

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