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MULTICOMPONENT ITERATIVE METHODS SOLVING STATIONARY PROBLEMS OF MATHEMATICAL PHYSICS

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Abstract. Additive iterative methods of complete approximation for stationary problems of mathematical physics are proposed. The convergence rate in the case of an arbitrary number of commutative and noncommutative partition operators is analysed. The optimal values of the iterative parameter are found and related estimates for the number of iterations are derived. Some applications of suggested iterative methods are discussed.

Key words: iterative algorithms, ADI method, multicomponent algorithms.

1 Introduction

The alternating direction implicit (ADI) method, suggested by Peaceman, Rachford and Douglas [10, 12], is widely used as iterative method for solving stationary problems of mathematical physics. At the present time there are many modifications of this method and schemes of its realization [11, 13, 14, 15]. It is known that the ADI method is based on special relaxation processes with a possibility of the reduction of the complicated problem to a sequence of more simple problems.

As a rule, it is assumed that the original operator is presented as a sum of two simpler operators. Many papers on the ADI method are based on this fact. In solving complicated problems of mathematical physics we often deal with the additive partition of the original operator on larger number of terms. We consider the representation

$$A = \sum_{\alpha=1}^{p} A_\alpha, \quad A_\alpha > 0, \quad \alpha = 1, 2, \ldots, p.$$

We are interested in the case $p > 2$ (the case $p = 2$ is considered in detail in the modern literature). The trivial extensions of the ADI type algorithms and methods of their investigation for $p > 2$ are not possible. In this paper we propose a new approach for solving this problem.

2 Statement of the Problem and Numerical Algorithms

Consider the operator equation

$$Ay = f, \tag{2.1}$$

where $A : H \to H$ is a linear operator (not necessarily discrete) acting in a real Hilbert space H with an inner product (u, v) and the norm $\|u\| = \sqrt{(u, u)}$. Suppose that A is a self-adjoint positive definite operator, i. e., $A = A^* \geqslant cE$, $c > 0$. Denote by H_A the space H equipped with the inner product $(u, v)_A = (Au, v)$ and the norm $\|u\|_A = \sqrt{(Au, u)}$.

It is well known that the Cauchy problem for the linear evolutionary equation

$$\frac{du}{dt} + Au = f, \quad t > 0, \quad u(0) = u_0, \tag{2.2}$$

has a solution $u(t)$ that converges in H to the solution of Eq. (2.1) as $t \to \infty$. The error $u(t) - y$ satisfies the inequality

$$\|u(t) - y\| \leqslant e^{-ct}\|u_0 - y\|. \tag{2.3}$$

The estimate (2.3) permits us to construct iterative methods for Eq. (2.1) with the use of various finite-difference schemes. Implicit schemes are optimal from the viewpoint of the stability and the convergence rate. For example, for the purely implicit difference scheme

$$\frac{y_{n+1} - y_n}{\tau} + Ay_{n+1} = f, \quad t_n = n\tau, \quad y_0 = u_0 \tag{2.4}$$

which approximates problem (2.2), we have the estimate

$$\|y_n - u\| \leqslant e^{-\delta t_n}\|y_0 - u\|, \quad \delta = \delta(\tau) > 0. \tag{2.5}$$

Therefore it seems natural to develop implicit economical iterative methods which have a similar convergence rate and, on the other hand, admit efficient implementations of the algorithm. A class of such methods is presented by additive difference schemes.

Additive methods are based on the representation of the operator A in the form $A = \sum_{\alpha=1}^{p} A_\alpha$ where each of the operators A_α is a stationary (possibly, degenerate) operator. Instead of equation (2.1) we consider the new equation written in the form $\sum_{\alpha=1}^{p} A_\alpha y_\alpha = f$. If $y = y_\alpha$, $\alpha = 1, 2, \ldots, p$, then this relation turns into (2.1).

To solve the nonstationary problem (2.2), one can use various difference schemes. If these schemes are asymptotically stable, then they can be used efficiently as iterative methods for stationary problem (2.1).

The alternating direction method was extended in [1], where the difference schemes

$$\frac{\widehat{y}_\alpha - y_\alpha}{\tau} + \sum_{\beta=1}^{\alpha} A_\alpha \widehat{y}_\beta + \sum_{\beta=\alpha+1}^{p} A_\beta y_\beta = f, \quad y_\alpha(0) = u_0, \quad \alpha = 1,2,\dots,p, \tag{2.6}$$

$$\frac{\widehat{y}_\alpha - y_\alpha}{\tau} + \sigma A_\alpha \big(\widehat{y}_\alpha - y_\alpha\big) + \sum_{\beta=1}^{p} A_\beta y_\beta = f, \quad y_\alpha(0) = u_0, \quad \alpha = 1,2,\dots,p, \tag{2.7}$$

were suggested for the approximation of problem (2.2). Schemes (2.6) and (2.7) are targeted for sequential and parallel computers, respectively.

Let us suppose that $f = f(t)$. The following theorem is valid (see, [5]).

Theorem 1. *If operators $A_\alpha \geqslant 0$, $\alpha = 1,2,\dots,p$, then the difference scheme* (2.6) *is stable with respect to the initial data and the right-hand side, and the solution admits the estimate*

$$\|y_\alpha\| \leqslant \|u_0\| + \|Au_0 - f(0)\| + M_1 t \max_t \|f_{\bar t}(t)\|, \tag{2.8}$$

where $M_1 > 0$ is a constant independent of τ.

Proof. Subtracting the neighbouring equations of the algorithm (2.6) one from another we get

$$y_{\alpha t} + \tau A_\alpha y_{\alpha t} = y_{\alpha-1,t}, \quad \alpha = 2,3,\dots,p,$$
$$y_{1t} + \tau A_1 y_{1t} = \check{y}_{pt} + \tau f_{\bar t}.$$

Considering the inner product of these equations with $y_{\alpha t}$ and taking into account conditions $A_\alpha \geqslant 0$, $\alpha = 1,2,\dots,p$, we prove that

$$\|y_{\alpha t}\| \leqslant \|\check{y}_{\alpha t}\| + \tau \|f_{\bar t}\|.$$

The estimate (2.8) follows trivially from this inequality. □

The similar theorem holds for the stability of the parallel algorithm (2.7) [2].

Theorem 2. *The difference scheme* (2.7) *is stable with respect to the initial data and the right-hand side if $\sigma \geqslant p/2$, $A_\alpha > 0$, $\alpha = 1,2,\dots,p$, and the solution admits the estimate*

$$\|y\|_1 \leqslant \Big(\|y(0)\|_1^2 + \Big\| \sum_{\alpha=1}^{p} A_\alpha y_\alpha(0) - f(0) \Big\|^2 + M_2 t \max_t \|f_{\bar t}(t)\|^2\Big)^{1/2}, \tag{2.9}$$

where $\|y\|_1 = \Big\| \sum_{\alpha=1}^{p} A_\alpha y_\alpha \Big\|$ and $M_2 > 0$ is a constant independent of τ.

As $\tau \to 0$, the solutions of both schemes under consideration converges to the solution of the original problem (2.2) at a rate $O(\tau)$.

Unfortunately, the analysis of the estimates (2.8) and (2.9) shows that the algorithms (2.6) and (2.7) cannot be efficiently used as iterative methods for solving Eq. (2.1). In particular, the convergence of the iterative vector-additive scheme

$$\frac{\overset{s+1}{y}_\alpha - \overset{s}{y}_\alpha}{\tau} + \sigma A_\alpha\big(\overset{s+1}{y}_\alpha - \overset{s}{y}_\alpha\big) + \sum_{\beta=1}^{p} A_\beta \overset{s}{y}_\beta = f, \quad \alpha = 1, 2, \ldots, p, \tag{2.10}$$

was proved in [6] for $\sigma \geqslant p/2$ without estimating the convergence rate (see, also [7, 9]). All components in (2.10) quite rapidly converge to each other, but the convergence to the original solution is rather slow. The linear combination of the components of the solution of system (2.10) is usually used for increasing of the convergence rate of the iterative method [6, 8, 14]. For example, as a solution of the original problem we can take averaged components of the discrete solution

$$\overset{s}{\widetilde{y}} = \sum_{\alpha=1}^{p} c_\alpha \overset{s}{y}_\alpha, \quad c_\alpha > 0, \ \sum_{\alpha=1}^{p} c_\alpha = 1.$$

Moreover, the high convergence rate is achieved if constants c_α coordinate with operators A_α, in particular, we can take $c_\alpha E = \Big(E + \sum_{\beta=1,\beta\neq\alpha}^{p} A_\alpha^{-1} A_\beta\Big)^{-1}$. Below we suggest the algorithms, which possess the high convergence rate and require no special choice of averaging of the components of the iterative method.

Consider two modified iterative multicomponent methods [3, 4]

$$\frac{\overset{s+1}{y}_\alpha - \overset{s}{y}{}^*_\alpha}{\tau} + \sum_{\beta=1}^{\alpha} A_\beta \overset{s+1}{y}_\beta + \sum_{\beta=\alpha+1}^{p} A_\beta \overset{s}{y}_\beta = f, \quad \alpha = 1, 2, \ldots, p, \tag{2.11}$$

$$\overset{s}{y}{}^*_1 = \overset{s}{y}_1, \quad \overset{s}{y}{}^*_\alpha = 0{,}5\big(\overset{s}{y}_\alpha + \overset{s}{y}_{\alpha-1}\big), \quad \alpha = 2, 3, \ldots, p,$$

$$\frac{\overset{s+1}{y}_\alpha - \overset{s}{\widetilde{y}}}{\tau} + \sigma A_\alpha\big(\overset{s+1}{y}_\alpha - \overset{s}{y}_\alpha\big) + \sum_{\beta=1}^{p} A_\beta \overset{s}{y}_\beta = f, \quad \alpha = 1, 2, \ldots, p, \tag{2.12}$$

$$\overset{s}{\widetilde{y}} = p^{-1} \sum_{\alpha=1}^{p} \overset{s}{y}_\alpha.$$

Below we show that the averaging of the solution in the first term of (2.11) and (2.12) provides a rapid convergence of all components to the averaged solution and the convergence of the same components to each other and to the solution of the original problem (2.1).

3 Convergence Analysis

Consider the stabilizing properties of the parallel method (2.12). This method can be classified as a regularized iterative finite-difference scheme of additive type. Summing relations (2.12) over $\alpha = 1, 2, \ldots, p$, we obtain that

$$\frac{\overset{s+1}{\widetilde{y}} - \overset{s}{\widetilde{y}}}{\tau} + \sigma^* \sum_{\alpha=1}^{p} A_\alpha \overset{s+1}{y}_\alpha + (1 - \sigma^*) \sum_{\alpha=1}^{p} A_\alpha \overset{s}{y}_\alpha = f, \qquad (3.1)$$

where $\sigma^* = \sigma/p$. The expression (3.1) approximates equation (2.2) with the first-order accuracy with respect to τ for $\sigma^* > 1/2$ and with the second-order accuracy for $\sigma^* = 1/2$. We can easily see that the iterative method (2.12) has a structure similar to that of the weighted difference scheme

$$\frac{\widehat{y} - y}{\tau} + \sigma^* A\widehat{y} + (1 - \sigma^*)Ay = f. \qquad (3.2)$$

The scheme (3.2) is asymptotically stable for $\tau < 2/\sqrt{c\Delta}$, $cE < A < \Delta E$, $\Delta > 0$ if $\sigma^* = 0{,}5$ and for any $\tau > 0$ if $\sigma^* > 0{,}5$. In general, for these values of σ, scheme (3.2) is the best multicomponent scheme for constructing iterative methods for solution of (2.1). The representation of (2.12) in the form (3.1) permits us to assume that the method (2.12) has asymptotic properties similar to those of (3.2). In this connection, it is interesting to study the convergence rate of the method for various values of parameter τ and to analyze the optimal choice of the iterative parameter.

The weighted scheme (3.2) belongs to the class of two-layer difference schemes [13], and its stability and convergence can easily be analysed with the use of the general theory based on the canonical form

$$By_t + Ay = f.$$

The expression (3.1) is a canonical form of the additive-averaged algorithm (2.12). It is similar to the scheme (3.2) only in appearance and it is a complete additive representation of scheme (2.12). However, it is impossible to use the expression (3.1) for solving equation (2.1). We consider the scheme (2.12) for this aim. The following assertion is valid.

Lemma 1. *If $A_\alpha \geqslant c_0 E$, $\alpha = 1, 2, \ldots, p$, $c_0 > 0$, then the iterative method (2.12) with $\sigma = p$ satisfies the inequality*

$$Q(\overset{s}{y}) \leqslant \Big(\frac{1}{q}\Big)^s Q(\overset{\circ}{y}), \qquad (3.3)$$

$$Q(\overset{s}{y}) = \|r(s)\|^2 + p^{-2}\tau^{-2}\|\overset{s}{v}\|_3^2, \qquad r(s) = \sum_{\alpha=1}^{p} A_\alpha \overset{s}{y}_\alpha - f,$$

$$\|\overset{s}{v}\|_3^2 = \sum_{\alpha,\beta=1,\alpha>\beta}^{p} \big(\overset{s}{v}{}^{(\alpha,\beta)}, \overset{s}{v}{}^{(\alpha,\beta)}\big), \qquad \overset{s}{v}{}^{(\alpha,\beta)} = \overset{s}{y}_\alpha - \overset{s}{y}_\beta, \quad q = 1 + 2c_0 p\tau.$$

Proof. Consider the inner product of equations (2.12) with $\tau A_\alpha \overset{s+1}{y}_{\alpha t}$ and let us add these relations over $\alpha = 1, 2, \ldots, p$:

$$\sum_{\alpha=1}^{p}\Big(\overset{s+1}{y}_\alpha - \overset{s}{\widetilde{y}}, A_\alpha \overset{s+1}{y}_{\alpha t}\Big) + p\tau^2 \sum_{\alpha=1}^{p} \Big\|A_\alpha \overset{s+1}{y}_{\alpha t}\Big\|^2 - 0{,}5\tau^2 \Big\|\sum_{\alpha=1}^{p} A_\alpha \overset{s+1}{y}_{\alpha t}\Big\|^2$$
$$+ 0{,}5\|r(s+1)\|^2 = 0{,}5\|r(s)\|^2. \qquad (3.4)$$

Let us transform the left-hand side of equation (3.4). We denote $\overset{s}{v}{}^{(\alpha,\beta)} = \overset{s}{y}_\alpha - \overset{s}{y}_\beta$. Then we can readily see that

$$\sum_{\alpha=1}^{p}\Big(\overset{s+1}{y}_\alpha - \overset{s}{\widetilde{y}}, A_\alpha \overset{s+1}{y}_{\alpha t}\Big) = \tau \sum_{\alpha=1}^{p}\Big(A_\alpha \overset{s+1}{y}_{\alpha t}, \overset{s+1}{y}_{\alpha t}\Big)$$

$$+ \frac{1}{p} \sum_{\alpha,\beta=1,\alpha>\beta}^{p} \Big(\overset{s}{v}{}^{(\alpha,\beta)}, A_\alpha \overset{s+1}{y}_{\alpha t} - A_\beta \overset{s+1}{y}_{\beta t}\Big). \qquad (3.5)$$

Since

$$A_\alpha \overset{s+1}{y}_{\alpha t} - A_\beta \overset{s+1}{y}_{\beta t} = -p^{-1}\tau^{-2}\overset{s+1}{v}{}^{(\alpha,\beta)},$$

it follows that the second term on the right-hand side of Eq. (3.5) is equal to

$$\frac{1}{p} \sum_{\alpha,\beta=1,\alpha>\beta}^{p} \Big(\overset{s}{v}{}^{(\alpha,\beta)}, A_\alpha \overset{s+1}{y}_{\alpha t} - A_\beta \overset{s+1}{y}_{\beta t}\Big) = -p^{-2}\tau^{-2} \sum_{\alpha,\beta=1,\alpha>\beta}^{p} \Big(\overset{s}{v}{}^{(\alpha,\beta)}, \overset{s+1}{v}{}^{(\alpha,\beta)}\Big)$$

$$= 0{,}5p^{-2}\tau^{-2}\Big(\tau^2\|\overset{s+1}{v}_t\|_3^2 - \|\overset{s+1}{v}\|_3^2 - \|\overset{s}{v}\|_3^2\Big),$$

where $\|\overset{s}{v}\|_3^2 = \sum_{\alpha,\beta=1,\alpha>\beta}^{p}(\overset{s}{v}{}^{(\alpha,\beta)}, \overset{s}{v}{}^{(\alpha,\beta)})$. Moreover, from the relation

$$0{,}5p\tau^2 \sum_{\alpha=1}^{p} \Big\|A_\alpha \overset{s+1}{y}_{\alpha t}\Big\|^2 - 0{,}5\tau^2 \Big\|\sum_{\alpha=1}^{p} A_\alpha \overset{s+1}{y}_{\alpha t}\Big\|^2 = 0{,}5p^{-2}\tau^{-2}\|\overset{s+1}{v}\|_3^2$$

we obtain the estimate

$$p\tau^2 \sum_{\alpha=1}^{p} \Big\|A_\alpha \overset{s+1}{y}_{\alpha t}\Big\|^2 - 0{,}5\tau^2 \Big\|\sum_{\alpha=1}^{p} A_\alpha \overset{s+1}{y}_{\alpha t}\Big\|^2 \geqslant p^{-2}\tau^{-2}\|\overset{s+1}{v}\|_3^2.$$

Taking into account the above transformations and the properties of operators A_α, from (3.4) we obtain that

$$c_0\tau \sum_{\alpha=1}^{p} \|\overset{s+1}{y}_{\alpha t}\|^2 + 0{,}5p^{-2}\tau^{-2}\|\overset{s+1}{v}\|_3^2 + 0{,}5p^{-2}\|\overset{s+1}{v}_t\|_3^2 + 0{,}5\|r(s+1)\|^2$$

$$\leqslant 0{,}5p^{-2}\tau^{-2}\|\overset{s}{v}\|_3^2 + 0{,}5\|r(s)\|^2. \qquad (3.6)$$

Since

$$\sum_{\alpha=1}^{p} \|\overset{s+1}{y}_{\alpha t}\|^2 = p\|\overset{s+1}{\widetilde{y}}_t\|^2 + p^{-1}\|\overset{s+1}{v}_t\|_3^2$$

and from expression (3.1) with $\sigma^* = 1$

$$\overset{s+1}{\widetilde{y}}_t = \frac{\overset{s+1}{\widetilde{y}} - \overset{s}{\widetilde{y}}}{\tau} = -r(s+1),$$

we have the following expression for the first term in inequality (3.6)

$$c_0\tau\sum_{\alpha=1}^{p}\|\overset{s+1}{y}_{\alpha t}\|^2=c_0\tau\Big(p\|r(s+1)\|^2+p^{-1}\|\overset{s+1}{v}_t\|_3^2\Big).$$

Representing the constant c_0 as the sum $c_0=\varepsilon_1+\varepsilon_2$, $\varepsilon_1,\varepsilon_2=\mathrm{const}>0$, with regard to the last relation, from (3.6) we obtain the inequality

$$\begin{aligned}0{,}5(1+2\varepsilon_1 p\tau)\|r(s+1)\|^2+\varepsilon_2\tau\sum_{\alpha=1}^{p}\|\overset{s+1}{y}_{\alpha t}\|^2+(0{,}5p^{-2}+\varepsilon_1 p^{-1}\tau)\|\overset{s+1}{v}_t\|_3^2\\+0{,}5p^{-2}\tau^{-2}\|\overset{s+1}{v}\|_3^2\leqslant 0{,}5p^{-2}\tau^{-2}\|\overset{s}{v}\|_3^2+0{,}5\|r(s)\|^2.\end{aligned}\tag{3.7}$$

To estimate $\sum_{\alpha=1}^{p}\|\overset{s+1}{y}_{\alpha t}\|^2$ we use the identity

$$\overset{s+1}{y}_{\alpha t}=-\Big(p\tau A_\alpha\overset{s+1}{y}_{\alpha t}+p^{-1}\tau^{-1}\sum_{\beta=1}^{p}(\overset{s}{y}_\alpha-\overset{s}{y}_\beta)+r(s)\Big),$$

which readily follows from (2.12). Considering the inner product of the last relation with itself and summing the resulting formula over $\alpha=1,2,\dots,p$, we obtain that

$$\begin{aligned}\sum_{\alpha=1}^{p}\|\overset{s+1}{y}_{\alpha t}\|^2=p^2\tau^2\sum_{\alpha=1}^{p}\|A_\alpha\overset{s+1}{y}_{\alpha t}\|^2+p^{-2}\tau^{-2}\sum_{\alpha=1}^{p}\Big\|\sum_{\beta=1}^{p}(\overset{s}{y}_\alpha-\overset{s}{y}_\beta)\Big\|^2+p\|r(s)\|^2\\-2\sum_{\alpha,\beta=1,\alpha>\beta}^{p}\Big(\overset{s}{v}^{(\alpha,\beta)},A_\alpha\overset{s+1}{y}_{\alpha t}-A_\beta\overset{s+1}{y}_{\beta t}\Big)+2p\tau\sum_{\alpha=1}^{p}\Big(A_\alpha\overset{s+1}{y}_{\alpha t},r(s)\Big)\\+\frac{2}{p\tau}\sum_{\alpha=1}^{p}\Big(\sum_{\beta=1}^{p}(\overset{s}{y}_\alpha-\overset{s}{y}_\beta),r(s)\Big).\end{aligned}$$

Since

$$\begin{aligned}p^2\tau^2\sum_{\alpha=1}^{p}\|A_\alpha\overset{s+1}{y}_{\alpha t}\|^2+2p\tau\sum_{\alpha=1}^{p}\Big(A_\alpha\overset{s+1}{y}_{\alpha t},r(s)\Big)+p\,\|r(s)\|^2=p^2\tau^2\sum_{\alpha=1}^{p}\|A_\alpha\overset{s+1}{y}_{\alpha t}\|^2\\-p\tau^2\Big\|\sum_{\alpha=1}^{p}A_\alpha\overset{s+1}{y}_{\alpha t}\Big\|^2+p\|r(s+1)\|^2=p^{-1}\tau^{-2}\|\overset{s+1}{v}\|_3^2+p\|r(s+1)\|^2,\end{aligned}$$

$$\begin{aligned}-2\sum_{\alpha,\beta=1,\alpha>\beta}^{p}\Big(\overset{s}{v}^{(\alpha,\beta)},A_\alpha\overset{s+1}{y}_{\alpha t}-A_\beta\overset{s+1}{y}_{\beta t}\Big)=2p^{-1}\tau^{-2}\sum_{\alpha,\beta=1,\alpha>\beta}^{p}\Big(\overset{s}{v}^{(\alpha,\beta)},\overset{s+1}{v}^{(\alpha,\beta)}\Big)\\=p^{-1}\tau^{-2}\Big(\|\overset{s+1}{v}\|_3^2+\|\overset{s}{v}\|_3^2-\tau^2\|\overset{s+1}{v}_t\|_3^2\Big),\end{aligned}$$

$$2p^{-1}\tau^{-1}\sum_{\alpha=1}^{p}\Big(\sum_{\beta=1}^{p}(\overset{s}{y}_\alpha-\overset{s}{y}_\beta),r(s)\Big)=0,$$

it follows from (3.7) that

$$(1+2c_0p\tau)\|r(s+1)\|^2+p^{-2}(1+2p(\varepsilon_1-\varepsilon_2)\tau)\|\overset{s+1}{v}_t\|_3^2+p^{-2}\tau^{-2}(1+4\varepsilon_2p\tau)\|\overset{s+1}{v}\|_3^2 \leqslant \|r(s)\|^2+p^{-2}\tau^{-2}\|\overset{s}{v}\|_3^2. \quad (3.8)$$

By setting $\varepsilon_2 = 0{,}5c_0$ in (3.8), we obtain the assertion of the lemma. □

Lemma 1 does not necessarily imply the convergence of the iterative scheme (2.12) to the solution of the problem (2.1) since the residual of the multicomponent discrete scheme $r(s)$ does not correlate with the natural residual $\overset{s}{r} = A\overset{s}{y}-f$ of the discretization. To estimate the convergence of the method (2.12), we introduce the error function $\overset{s}{\rho} = \overset{s}{\widetilde{y}} - y$. Using the identity

$$\overset{s}{y}_\alpha = \overset{s}{\widetilde{y}} + \frac{1}{p}\sum_{\beta=1}^{p}(\overset{s}{y}_\alpha - \overset{s}{y}_\beta) = \overset{s}{\widetilde{y}} + \frac{1}{p}\sum_{\beta=1}^{p}\overset{s}{v}^{(\alpha,\beta)}$$

and the expression for the residual $r(s)$, we obtain that

$$A\overset{s}{\rho} = -\sum_{\alpha=1}^{p}A_\alpha\Big(\frac{1}{p}\sum_{\beta=1}^{p}\overset{s}{v}^{(\alpha,\beta)}\Big) + r(s),$$

whence it follows that

$$\overset{s}{\rho} = -\sum_{\alpha=1}^{p}B_\alpha\Big(\frac{1}{p}\sum_{\beta=1}^{p}\overset{s}{v}^{(\alpha,\beta)}\Big) + A^{-1}r(s),$$

where $B_\alpha = A^{-1}A_\alpha = \Big(E + \sum_{\beta=1,\beta\neq\alpha}^{p} A_\alpha^{-1}A_\beta\Big)^{-1}$. If operators A_α, $\alpha = 1,2,\dots,p$, are pair-wise commuting, then, obviously, $\|B_\alpha\| < 1$ and

$$\|\overset{s}{\rho}\| \leqslant c^{-1}\|r(s)\| + \|\overset{s}{v}\|_3.$$

From (3.3) we obtain

$$\|\overset{s}{v}\|_3 \leqslant p\tau\Big(\Big(\frac{1}{q}\Big)^s Q(\overset{\circ}{y})\Big)^{1/2}, \quad c^{-1}\|r(s)\| \leqslant c^{-1}\Big(\Big(\frac{1}{q}\Big)^s Q(\overset{\circ}{y})\Big)^{1/2}. \quad (3.9)$$

Summing up these inequalities, we get the estimate

$$\|\overset{s}{\rho}\| \leqslant (p\tau + c^{-1})q^{-s/2}\Big(Q(\overset{\circ}{y})\Big)^{1/2}.$$

Hence the following theorem holds.

Theorem 3. *Let $A_\alpha \geqslant c_0E$, $\alpha = 1,2,\dots,p$, $c_0 > 0$, and the operators A_α be pair-wise commuting. Then the additive iterative method* (2.12) *with $\sigma = p$ converges to the solution of* (2.1) *and the convergence rate can be estimated as*

$$\big\|\overset{s}{\widetilde{y}} - y\big\| \leqslant c^{-1}\|r(s)\| + \|\overset{s}{v}\|_3 \leqslant \frac{p\tau + c^{-1}}{(1+2c_0p\tau)^{s/2}}\|r(0)\|^{1/2}. \quad (3.10)$$

The iterative method (2.12) is convergent for any $\tau > 0$, but, by (3.9), the optimal value of the iterative parameter is attained at $\tau = \tau_0 \sim p^{-1}c^{-1}$. Therefore, the convergence rate of the additive iterative method (2.12) depends only on the lower boundary of the spectrum of the operators A, A_α; i.e., the estimate (3.10) is similar to the convergence rate estimate (2.5) for a purely implicit difference scheme (2.4).

Using Theorem 3, we can estimate the number s of iterations needed to reduce the original error by a factor of $1/\varepsilon$. For this purpose it suffices to require that $(p\tau + c^{-1})(1 + 2c_0 p\tau)^{-s/2} \leqslant \varepsilon$ and take into account the condition $c = pc_0$. Then for $\tau = p^{-1}c^{-1}$ we have

$$s \geqslant s_0(\varepsilon) = \frac{2\ln(2c^{-1}/\varepsilon)}{\ln(1 + 2p^{-1})}.$$

To estimate the convergence rate of the iterative method (2.12) in the case of noncommutative operators A_α, one can use the following assertion.

Theorem 4. *Let $c_0 E \leqslant A_\alpha \leqslant \Delta_0 E$, $\alpha = 1, 2, \ldots, p$, $c_0 > 0$, and $\Delta_0 > 0$. Then the additive iterative method* (2.12) *with $\sigma = p$ converges to the solution of Eq.* (2.1) *and the convergence rate can be estimated as*

$$\big\|\overset{s}{\tilde{y}} - y\big\|_A \leqslant c^{-1/2}\|r(s)\| + \Big(\sum_{\alpha=1}^{p}\Big\|\frac{1}{p}\sum_{\beta=1}^{p}\overset{s}{v}{}^{(\alpha,\beta)}\Big\|_{A_\alpha}^2\Big)^{1/2} \leqslant \big(c^{-1/2} + \tau\Delta^{1/2}\big)q^{-s/2}\|r(0)\|^{1/2}, \quad (3.11)$$

where $\Delta = p\Delta_0$.

The estimate (3.11) involves the term $\tau\Delta^{1/2}$ that contains the upper boundary of the spectrum of the operator A. Thus, one needs to perform the preliminary discretization of the original problem and to connect the iterative parameter and the space grid step by the relation $\tau \sim c^{-1/2}\Delta^{-1/2}$. In this case the estimate for the number of iterations gets the form

$$s \geqslant s_0(\varepsilon) = \frac{2\ln(2c^{-1/2}/\varepsilon)}{\ln(1 + 2c^{1/2}\Delta^{-1/2})}.$$

The resulting convergence condition coincides with the constraint imposed on τ in the classical alternating direction method. Moreover, the dependence of the number of iterations of the method on the discretization step h is given by the formula $s_0(\varepsilon) = O(\ln h^{-1})$.

Note that if $\sigma > p/2$ (i. e., $\sigma^* > 0{,}5$), then Theorems 3 and 4 remain valid. For $\sigma = p/2$ the convergence of the iterative process (2.12) can be proved only for bounded operators A_α, i. e., for finite-difference or projection-difference schemes.

Now we investigate in detail the convergence of the sequential iterative method (2.11). We assume that A_α, $\alpha = 2, 3, \ldots, p$, are nonnegative operators and the operator A_1 is positive definite, i.e.,

$$(A_1 y, y) \geqslant c_0\|y\|^2, \quad c_0 = \text{const} > 0, \quad A_\alpha \geqslant 0, \quad \alpha = 2, 3, \ldots, p.$$

Considering the inner product of Eq. (2.11) with $\tau A_\alpha \overset{s+1}{y}_{\alpha t}$ and summing up the resulting relations over $\alpha = 1, 2, \ldots, p$, we obtain

$$\sum_{\alpha=1}^{p}\Bigl(\overset{s+1}{y}_\alpha - \overset{s}{y}{}^*_\alpha, A_\alpha \overset{s+1}{y}_{\alpha t}\Bigr) + 0{,}5\tau^2 \sum_{\alpha=1}^{p}\Bigl\|A_\alpha \overset{s+1}{y}_{\alpha t}\Bigr\|^2 + 0{,}5\|r(s+1)\|^2 = 0{,}5\|r(s)\|^2, \tag{3.12}$$

where $\overset{s+1}{y}_{\alpha t} = (\overset{s+1}{y}_\alpha - \overset{s}{y}_\alpha)/\tau$. Setting $\overset{s}{v}{}^{(\alpha,\alpha-1)} = \overset{s}{y}_\alpha - \overset{s}{y}_{\alpha-1}$, we rewrite the first term on the left-hand side in (3.12) in the form

$$\sum_{\alpha=1}^{p}\Bigl(\overset{s+1}{y}_\alpha - \overset{s}{y}{}^*_\alpha, A_\alpha \overset{s+1}{y}_{\alpha t}\Bigr) = \tau \sum_{\alpha=1}^{p}\Bigl(A_\alpha \overset{s+1}{y}_{\alpha t}, \overset{s+1}{y}_{\alpha t}\Bigr) - \frac{1}{2}\sum_{\alpha=2}^{p}\Bigl(\overset{s}{v}{}^{(\alpha,\alpha-1)}, A_\alpha \overset{s+1}{y}_{\alpha t}\Bigr). \tag{3.13}$$

Subtracting the equations in (2.11) with indices α and $\alpha-1$ from each other, we arrive at the relation $\tau^{-2}\overset{s+1}{w}{}^{(\alpha)} = A_\alpha \overset{s+1}{y}_{\alpha t}$, where

$$\overset{s+1}{w}{}^{(2)} = \overset{s+1}{v}{}^{(2,1)} - 0{,}5\,\overset{s}{v}{}^{(2,1)},$$

$$\overset{s+1}{w}{}^{(\alpha)} = \overset{s+1}{v}{}^{(\alpha,\alpha-1)} - 0{,}5\Bigl(\overset{s}{v}{}^{(\alpha,\alpha-1)} + \overset{s}{v}{}^{(\alpha-1,\alpha-2)}\Bigr), \quad \alpha = 3, 4, \ldots, p.$$

Furthermore, we have the obvious identity for $\alpha = 3, 4, \ldots, p$:

$$\overset{s+1}{v}{}^{(\alpha,\alpha-1)} = 0{,}5\Bigl(\bigl(\overset{s+1}{v}{}^{(\alpha,\alpha-1)} + \overset{s+1}{v}{}^{(\alpha-1,\alpha-2)}\bigr) + \bigl(\overset{s+1}{v}{}^{(\alpha,\alpha-1)} - \overset{s+1}{v}{}^{(\alpha-1,\alpha-2)}\bigr)\Bigr).$$

Using these relations, we represent the second term on the right-hand side in (3.13) as the sum of two terms:

$$-0{,}5\Bigl(\overset{s}{v}{}^{(2,1)}, A_2 \overset{s+1}{y}_{2t}\Bigr) = 0{,}5\tau^{-2}\Bigl(\overset{s}{v}{}^{(2,1)}, \overset{s+1}{v}{}^{(2,1)}\Bigr) - 0{,}25\tau^{-2}\|\overset{s}{v}{}^{(2,1)}\|^2,$$

$$-0{,}5\sum_{\alpha=3}^{p}\Bigl(\overset{s}{v}{}^{(\alpha,\alpha-1)}, A_\alpha \overset{s+1}{y}_{\alpha t}\Bigr) = S_1 + S_2,$$

where

$$S_{1,2} = 0{,}25\tau^{-2}\sum_{\alpha=3}^{p}\bigl(\overset{s+1}{w}{}^{(\alpha)}, \overset{s}{v}{}^{(\alpha,\alpha-1)} \pm \overset{s}{v}{}^{(\alpha-1,\alpha-2)}\bigr).$$

We rewrite S_1 as follows:

$$S_1 = 0{,}25\tau^{-2}\sum_{\alpha=3}^{p}\Bigl(\overset{s+1}{v}{}^{(\alpha,\alpha-1)}, \overset{s}{v}{}^{(\alpha,\alpha-1)} + \overset{s}{v}{}^{(\alpha-1,\alpha-2)}\Bigr) - (0{,}5)^3\tau^{-2}\sum_{\alpha=3}^{p}\Bigl\|\overset{s}{v}{}^{(\alpha,\alpha-1)} + \overset{s}{v}{}^{(\alpha-1,\alpha-2)}\Bigr\|^2.$$

For S_2, we have the estimate

$$|S_2| \leqslant 0{,}25\tau^{-2}\sum_{\alpha=3}^{p}\Bigl(\overset{s+1}{w}{}^{(\alpha)}, \overset{s+1}{w}{}^{(\alpha)}\Bigr) + (0{,}25)^2\tau^{-2}\sum_{\alpha=3}^{p}\Bigl\|\overset{s}{v}{}^{(\alpha,\alpha-1)} - \overset{s}{v}{}^{(\alpha-1,\alpha-2)}\Bigr\|^2.$$

We represent the second term on the left-hand side in (3.12) as

$$0{,}5\tau^2 \sum_{\alpha=1}^{p} \Big\| A_\alpha \overset{s+1}{y}_{\alpha t} \Big\|^2 = 0{,}5\tau^2 \| A_1 \overset{s+1}{y}_{1t} \|^2 + 0{,}5\tau^{-2} \Big(\overset{s+1}{w}{}^{(2)}, \overset{s+1}{w}{}^{(2)} \Big) + 0{,}5\tau^{-2} \sum_{\alpha=3}^{p} \Big(\overset{s+1}{w}{}^{(\alpha)}, \overset{s+1}{w}{}^{(\alpha)} \Big). \quad (3.14)$$

To estimate the inner products occurring in (3.14), we use the relations

$$0{,}5\tau^{-2}\Big(\overset{s+1}{w}{}^{(2)}, \overset{s+1}{w}{}^{(2)}\Big) = 0{,}5\tau^{-2}\Big(\|\overset{s+1}{v}{}^{(2,1)}\|^2 - (\overset{s+1}{v}{}^{(2,1)}, \overset{s}{v}{}^{(2,1)}) + 0{,}25\|\overset{s}{v}{}^{(2,1)}\|^2\Big),$$

$$0{,}5\tau^{-2}\Big(\overset{s+1}{w}{}^{(\alpha)}, \overset{s+1}{w}{}^{(\alpha)}\Big) = 0{,}25\tau^{-2}\Big(\|\overset{s+1}{w}{}^{(\alpha)}\|^2 + \|\overset{s+1}{v}{}^{(\alpha,\alpha-1)}\|^2 + 0{,}25\|\overset{s}{v}{}^{(\alpha,\alpha-1)}\|^2 - (\overset{s+1}{v}{}^{(\alpha,\alpha-1)}, \overset{s}{v}{}^{(\alpha,\alpha-1)} + \overset{s}{v}{}^{(\alpha-1,\alpha-2)})\Big), \quad \alpha = 3, 4, \ldots, p.$$

Using estimates given above, we get from (3.12) the following inequality

$$\tau \sum_{\alpha=1}^{p} \Big(A_\alpha \overset{s+1}{y}_{\alpha t}, \overset{s+1}{y}_{\alpha t}\Big) + 0{,}5\tau^2\|A_1 \overset{s+1}{y}_{1t}\|^2 + 0{,}5\tau^{-2}\|\overset{s+1}{v}{}^{(2,1)}\|^2 + (1/8)\tau^{-2}\|\overset{s+1}{v}\|^2 + 0{,}5\|r(s+1)\|^2 \leqslant (1/8)\tau^{-2}\|\overset{s}{v}{}^{(2,1)}\|^2 + (1/16)\tau^{-2}\|\overset{s}{v}\|^2 + 0{,}5\|r(s)\|^2, \quad (3.15)$$

where $\|\overset{s}{v}\|^2 = \sum_{\alpha=3}^{p}\big(\overset{s}{v}{}^{(\alpha,\alpha-1)}, \overset{s}{v}{}^{(\alpha,\alpha-1)}\big)$.

Further, from Eq. (2.11) we obtain the relation

$$\overset{s+1}{y}_{1t} = -\Big(\tau A_1 \overset{s}{y}_{1t} + r(s)\Big),$$

which, together with the property $\Big(A_1\overset{s}{y}_{1t}, \overset{s}{y}_{1t}\Big) \geqslant c_0\|\overset{s}{y}_{1t}\|^2$ of the operator A, implies that

$$c_0\tau\|\overset{s+1}{y}_{1t}\|^2 = c_0\tau\Big(\tau^2\|A_1 \overset{s+1}{y}_{1t}\|^2 + 2\tau(A_1\overset{s}{y}_{1t}, r(s)) + \|r(s)\|^2\Big).$$

Using the ε-inequality with $\varepsilon = 1/4$ we estimate the second term on the right-hand side in the last relation

$$2\tau\Big(A_1\overset{s}{y}_{1t}, r(s)\Big) \leqslant 0{,}5\Big(\|r(s)\|^2 + 2\tau^2\|A_1\overset{s}{y}_{1t}\|^2\Big),$$

and from (3.15) with $0{,}5 - c_0\tau \geqslant 0$, we finally obtain the relation

$$\tau^{-2}\Big(\|\overset{s+1}{v}{}^{(2,1)}\|^2 + 0{,}25\|\overset{s+1}{v}\|^2\Big) + \|r(s+1)\|^2 \leqslant 0{,}5\tau^{-2}\Big(\|\overset{s}{v}{}^{(2,1)}\|^2 + 0{,}25\|\overset{s}{v}\|^2\Big) + (1 - c_0\tau)\|r(s)\|^2.$$

The last inequality implies the following assertion.

Lemma 2. *Let* $0{,}5 - c_0\tau \geqslant 0$, A_1 *be a positive definite operator, i.e.,* $A_1 \geqslant c_0 E$, *and let* A_α, $\alpha = 2, 3, \ldots, p$ *be nonnegative operators. Then the iterative process* (2.11) *satisfies the estimate*

$$\overset{s}{Q} \leqslant q^s \overset{\circ}{Q}, \quad s = 1, 2, \ldots, \tag{3.16}$$

where $\overset{s}{Q} = \|r(s)\|^2 + \tau^{-2}\Big(\|\overset{s}{v}{}^{(2,1)}\|^2 + 0{,}25\|\overset{s}{v}\|^2\Big)$ *and* $q = \max\{0.5, 1 - c_0\tau\}$.

It follows from (3.16) that $\overset{s}{Q} \to 0$ as $s \to \infty$. The optimal rate of the convergence to zero is attained at $c_0\tau = 0{,}5$.

Just as in the case of the parallel method (2.12), Lemma 2 is not sufficient to prove the convergence of the iterative method (2.11). To estimate the convergence in standard norms, we introduce the error function $\overset{s}{\rho} = \overset{s}{y}_1 - y$. Note that, in general, instead of $\overset{s}{y}_1$, we can choose an arbitrary $\overset{s}{y}_\alpha$ ($\alpha = 1, 2, \ldots, p$) or their arbitrary average $\overset{s}{\widetilde{y}} = p^{-1}\sum_{\alpha=1}^{p} \overset{s}{y}_\alpha$. Using the identity

$$\overset{s}{y}_\alpha = \overset{s}{y}_1 + \sum_{\beta=2}^{\alpha}(\overset{s}{y}_\beta - \overset{s}{y}_{\beta-1}) = \overset{s}{y}_1 + \sum_{\beta=2}^{\alpha} \overset{s}{v}{}^{(\beta,\beta-1)}$$

we can readily obtain the relation

$$\overset{s}{\rho} = -\sum_{\alpha=2}^{p} B_\alpha \sum_{\beta=2}^{\alpha} \overset{s}{v}{}^{(\beta,\beta-1)} + A^{-1}r(s), \tag{3.17}$$

where $B_\alpha = A^{-1}A_\alpha = \Big(E + \sum_{\beta=1,\beta\neq\alpha}^{p} A_\alpha^{-1}A_\beta\Big)^{-1}$. If the operators A_α, $\alpha = 1, 2, \ldots, p$, pair-wise commute, then $\|B_\alpha\| < 1$ and inequality (3.17) implies the estimate

$$\|\overset{s}{\rho}\| \leqslant c^{-1}\|r(s)\| + p\|\overset{s}{v}\|, \quad cE \leqslant A. \tag{3.18}$$

It follows from (3.16) that $\|\overset{s}{v}\| \leqslant 4\tau\Big(q^s\overset{\circ}{Q}\Big)^{1/2}$ and $c^{-1}\|r(s)\| \leqslant c^{-1}\Big(q^s\overset{\circ}{Q}\Big)^{1/2}$, which, together with (3.18) gives the estimate $\|\overset{s}{\rho}\| \leqslant (4p\tau + c^{-1})\Big(q^s\overset{\circ}{Q}\Big)^{1/2}$. We have thereby proved the following assertion.

Theorem 5. *Let the assumptions of Lemma 2 be satisfied, and let the operators* A_α, $\alpha = 1, 2, \ldots, p$, *be pair-wise commuting. Then the iterative method* (2.11) *converges to the solution of the original equation* (2.1), *and its convergence rate can be estimated as*

$$\|\overset{s}{\rho}\| \leqslant (4p\tau + c^{-1})q^{s/2}\|r(0)\|. \tag{3.19}$$

From (3.19), we can obtain an estimate for the number of iterations required to achieve a given accuracy ε:

$$s \geqslant s_0 = \frac{2\ln\Big((4p\tau + c^{-1})/\varepsilon\Big)}{\ln q^{-1}}. \tag{3.20}$$

If the condition of pair-wise commutativity of the operators A_α fails, then, using (3.17), we can readily justify the following assertion.

Theorem 6. *Let the assumptions of Lemma 2 be satisfied. Then the iterative process* (2.11) *is convergent, and its error can be estimated as*

$$\|\overset{s}{\rho}\|_A \leqslant \frac{1}{\sqrt{c}}\|r(s)\| + \Big(\sum_{\alpha=1}^{p}\Big\|\sum_{\beta=2}^{p}\overset{s}{v}{}^{(\beta,\beta-1)}\Big\|_{A_\alpha}\Big)^{1/2} \leqslant \Big(\frac{1}{\sqrt{c}} + 4p\Delta^{1/2}\tau\Big)q^{s/2}\|r(0)\|^{1/2},$$

$$s \geqslant s_0 = \frac{2\ln\Big((c^{-1/2} + 4p\Delta^{1/2}\tau)/\varepsilon\Big)}{\ln q^{-1}}, \quad \Delta E \geqslant A.$$

It follows from (3.19) and (3.20) that, in commutative case, the convergence rate of the sequential method (2.11) (just as of the parallel additive method (2.12)) depends only on the iterative parameter τ and the lower boundary of the spectrum of the operator A and A_α.

4 Conclusive Remarks

The multicomponent iterative methods suggested above for solution of stationary problems are additive in form as well as in the implementation, in contrast to classical splitting-up methods [11, 15], factorization methods [13], and the alternating direction method, which are based on additive approximation principles. All these methods are related to the sequential realization of the solution finding process, and, in addition to the partition additivity, we have a multiplicative form of an algorithm. Moreover, the suggested approach for constructing economical iterative methods for equations with unbounded operators seems to be quite useful for elaborating new algorithms to solve such problems. From our viewpoint, algorithms (2.11) and (2.12) are most effective for iterative directional partition methods. In the noncommutative case the advantages of these methods is that they can be applied for problems of arbitrary dimensions. The classical alternating direction method and its various modifications can be applied for solution of two-dimensional problems only.

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