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POSITIVE SOLUTIONS FOR SYSTEMS OF M-POINT NONLINEAR BOUNDARY VALUE PROBLEMS

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Abstract. Positive solutions $(u(t), v(t))$ are sought for the nonlocal (m-point) nonlinear system of boundary value problems, $u'' + \lambda a(t)f(v) = 0,\ v'' + \lambda b(t)g(u) = 0$, for $0 < t < 1$, and satisfying, $u(0) = 0, u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), v(0) = 0, v(1) = \sum_{i=1}^{m-2} a_i v(\xi_i)$. An application of a Guo-Krasnosel'skii fixed point theorem yields sufficient values of λ for which such positive solutions exist.

Key words: nonlocal (m-point) boundary value problem, system of differential equations, eigenvalue problem, positive solutions.

1 Introduction

We want to determine such values of the parameter λ, that the system of nonlocal (m-point) boundary value problems,

$$\begin{aligned} &u''(t) + \lambda a(t)f(v(t)) = 0, \quad 0 < t < 1, \\ &v''(t) + \lambda b(t)g(u(t)) = 0, \quad 0 < t < 1, \end{aligned} \tag{1.1}$$

$$\begin{aligned} &u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), \\ &v(0) = 0, \quad v(1) = \sum_{i=1}^{m-2} a_i v(\xi_i), \end{aligned} \tag{1.2}$$

with $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1$, $a_i \geq 0$ for $i = 1, 2, \ldots, m-3$, $a_{m-2} > 0$, and $\sum_{i=1}^{m-2} a_i \xi_i < 1$, has a solution $(u(t), v(t))$, such that $u(t) > 0$ and $v(t) > 0$ for $0 < t < 1$.

In addition, we put the following assumptions on the functions f and g:

(A) $f, g : [0, \infty) \to [0, \infty)$ are continuous,

(B) $a, b : [0, 1] \to [0, \infty)$ are continuous, and given any $[c, d] \subset [0, 1]$, there exist $t_1, t_2 \in [c, d]$ such that $a(t_1) > 0$ and $b(t_2) > 0$,

(C) all limits

$$f_0 := \lim_{x \to 0^+} \frac{f(x)}{x},\ g_0 := \lim_{x \to 0^+} \frac{g(x)}{x},\ f_\infty := \lim_{x \to \infty} \frac{f(x)}{x},\ g_\infty := \lim_{x \to \infty} \frac{g(x)}{x}$$

exist as positive real numbers.

The importance of positive solutions for boundary value problems, both theoretically as well as from the perspective of their applications in physical and engineering sciences, has been well documented in the literature; see, for example, [1, 5, 6, 7, 9, 12, 13, 14, 16, 20, 27]. While many of these referenced papers have been devoted to scalar problems, there is much emerging interest in boundary value problems for systems of differential equations [10, 11, 18, 22, 26, 28], and a good deal of research has also involved positive solutions for multipoint nonlinear eigenvalue problems in both scalar and systems contexts [2, 8, 18, 23]. In this paper, we extend some of the results obtained in [2] for the system (1.1)–(1.2). Again, the main tool relied upon is the Guo-Krasnosel'skii fixed point theorem for operators leaving a Banach space cone invariant [7].

2 Some Preliminaries

In this section, we state some preliminary lemmas and the well-known Guo-Krasnosel'skii fixed point theorem.

Lemma 1. [25] *Let $a_i \geq 0$ for $i = 1, 2, \ldots, m-2$ and $\sum_{i=1}^{m-2} a_i \xi_i \neq 1$. Then for any $y \in C[0, 1]$ the boundary value problem*

$$u''(t) + y(t) = 0, \quad 0 < t < 1 \tag{2.1}$$

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), \tag{2.2}$$

has a unique solution given by

$$\begin{aligned} u(t) = &- \int_0^1 (t-s) y(s) ds + \frac{t}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \int_0^1 (1-s) y(s) ds \\ &+ \frac{t}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s) y(s) ds. \end{aligned} \tag{2.3}$$

We mention that the authors in papers [3, 4, 24] have obtained more general solvability results than those in Lemma 1.

From (2.3) we notice that if $y \geq 0$ and $\sum_{i=1}^{m-2} a_i\xi_i < 1$ (see [25])

$$u(t) \leq \frac{t}{1-\sum_{i=1}^{m-2} a_i\xi_i} \int_0^1 (1-s)y(s)ds, \quad 0 \leq t \leq 1, \tag{2.4}$$

and

$$u(1) \geq \frac{1}{1-\sum_{i=1}^{m-2} a_i\xi_i} \sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^1 \xi_i(1-s)y(s)ds. \tag{2.5}$$

Lemma 2. [20] *Let* $a_i \geq 0$ *for* $i = 1, 2, \ldots, m-2$ *and* $\sum_{i=1}^{m-2} a_i\xi_i < 1$. *If* $y \in C[0,1]$ *and* $y \geq 0$, *then the unique solution* u *of* (2.1), (2.2) *satisfies*

$$\inf_{t\in[\xi_{m-2},1]} u(t) \geq \gamma\|u\|,$$

where

$$\gamma = \min\left\{\frac{a_{m-2}(1-\xi_{m-2})}{1-a_{m-2}\xi_{m-2}}, a_{m-2}\xi_{m-2}, \xi_1\right\}. \tag{2.6}$$

Lemma 3. [17] *Suppose* $0 < \sum_{i=1}^{m-2} a_i\xi_i < 1$. *The Green's function for the boundary value problem*

$$-y''(t) = 0, \qquad 0 < t < 1 \tag{2.7}$$

$$y(0) = 0, \quad y(1) = \sum_{i=1}^{m-2} a_i y(\xi_i) \tag{2.8}$$

is given by

$$G(t,s) = \begin{cases} \dfrac{s(1-t) - \sum_{i=1}^{m-2} a_i(\xi_i - t)s + \sum_{i=1}^{m-2} a_i\xi_i(t-s)}{1-\sum_{i=1}^{m-2} a_i\xi_i}, \\ \quad 0 \leq t \leq 1, \quad \xi_{i-1} \leq s \leq \min\{\xi_i, t\}, \quad i = 1, 2, \ldots, m-1; \\ \dfrac{t\left[(1-s) - \sum_{i=1}^{m-2} a_i(\xi_i - s)\right]}{1-\sum_{i=1}^{m-2} a_i\xi_i}, \\ \quad 0 \leq t \leq 1, \quad \max\{\xi_{i-1}, t\} \leq s \leq \xi_i, \quad i = 1, 2, \ldots, m-1. \end{cases}$$

We note that a pair $(u(t), v(t))$ is a solution of the eigenvalue problem (1.1)–(1.2) if, and only if,

$$u(t) = \lambda \int_0^1 G(t,s)a(s)f\left(\lambda \int_0^1 G(s,r)b(r)g(u(r))dr\right)ds, \quad 0 \le t \le 1,$$

and

$$v(t) = \lambda \int_0^1 G(t,s)b(s)g(u(s))ds, \quad 0 \le t \le 1.$$

Values of λ for which there are positive solutions (positive with respect to a cone) of (1.1)–(1.2) will be determined via applications of the following fixed point theorem.

Theorem 1. *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume Ω_1 and Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$T : \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}$$

be a completely continuous operator such that, either

(i) $\|Tu\| \le \|u\|, u \in \mathcal{P} \cap \partial\Omega_1,$ *and* $\|Tu\| \ge \|u\|, u \in \mathcal{P} \cap \partial\Omega_2,$ *or*

(ii) $\|Tu\| \ge \|u\|, u \in \mathcal{P} \cap \partial\Omega_1,$ *and* $\|Tu\| \le \|u\|, u \in \mathcal{P} \cap \partial\Omega_2.$

Then T has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 Positive Solutions in a Cone

In this section, we apply Theorem 1 to obtain solutions in a cone (that is, positive solutions) of (1.1)–(1.2). For our construction, let $\mathcal{B} = C[0,1]$ with supremum norm, $\|\cdot\|$, and define a cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \left\{ x \in \mathcal{B} \mid x(t) \ge 0 \text{ on } [0,1], \text{ and } \min_{t \in [\xi_{m-2},\, 1]} x(t) \ge \gamma \|x\| \right\}.$$

For our first result, define positive numbers L_1 and L_2 by

$$L_1 : \;= \frac{1 - \sum_{i=1}^{m-2} a_i\xi_i}{\gamma^2} \max\left\{ \Big[\sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^1 \xi_i(1-r)a(r)f_\infty dr \Big]^{-1}, \Big[\sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^1 \xi_i(1-r)b(r)g_\infty dr \Big]^{-1} \right\},$$

and

$$L_2 := \Big(1 - \sum_{i=1}^{m-2} a_i\xi_i\Big) \min\left\{ \Big[\int_0^1 (1-r)a(r)f_0 dr \Big]^{-1}, \Big[\int_0^1 (1-r)b(r)g_0 dr \Big]^{-1} \right\}.$$

Theorem 2. *Assume conditions* (A), (B) *and* (C) *are satisfied. Then, for each* λ *satisfying*

$$L_1 < \lambda < L_2, \tag{3.1}$$

there exists a pair (u, v) *satisfying* (1.1)–(1.2) *such that* $u(t) > 0$ *and* $v(t) > 0$ *on* $(0, 1)$.

Proof. Let λ be as in (3.1), and let $\epsilon > 0$ be chosen such that

$$\frac{1-\sum_{i=1}^{m-2} a_i\xi_i}{\gamma^2} \max\Big\{\Big[\sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^{1} \xi_i(1-r)a(r)(f_\infty - \epsilon)dr\Big]^{-1}, \Big[\sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^{1} \xi_i(1-r)b(r)(g_\infty - \epsilon)dr\Big]^{-1}\Big\} \le \lambda$$

and

$$\lambda \le \Big(1 - \sum_{i=1}^{m-2} a_i\xi_i\Big) \min\Big\{\Big[\int_0^1 (1-r)a(r)(f_0+\epsilon)dr\Big]^{-1}, \Big[\int_0^1 (1-r)b(r)(g_0+\epsilon)dr\Big]^{-1}\Big\}.$$

Define the integral operator $T : \mathcal{P} \to \mathcal{B}$ by

$$Tu(t) := \lambda \int_0^1 G(t,s)a(s)f\left(\lambda \int_0^1 G(s,r)b(r)g(u(r))dr\right)ds, \quad u \in \mathcal{P}. \tag{3.2}$$

We seek suitable fixed points of T in the cone $\mathcal{P}$. By Lemma 2, $T\mathcal{P} \subset \mathcal{P}$. In addition, standard arguments show that T is completely continuous. Now, from the definitions of f_0 and g_0, there exists an $H_1 > 0$ such that

$$f(x) \le (f_0 + \epsilon)x \text{ and } g(x) \le (g_0 + \epsilon)x, \quad 0 < x \le H_1.$$

Let $u \in \mathcal{P}$ with $\|u\| = H_1$. We first have from (2.4) and choice of ϵ,

$$\begin{aligned}\lambda \int_0^1 G(s,r)b(r)g(u(r))dr &\le \lambda \frac{t}{1-\sum_{i=1}^{m-2} a_i\xi_i} \int_0^1 (1-r)b(r)g(u(r))dr \\ &\le \lambda \frac{1}{1-\sum_{i=1}^{m-2} a_i\xi_i} \int_0^1 (1-r)b(r)(g_0+\epsilon)u(r)dr \\ &\le \lambda \frac{1}{1-\sum_{i=1}^{m-2} a_i\xi_i} \int_0^1 (1-r)b(r)dr(g_0+\epsilon)\|u\| \le \|u\| = H_1.\end{aligned}$$

As a consequence, using again (2.4), and the choice of ϵ, we have

$$\begin{aligned} Tu(t) &= \lambda \int_0^1 G(t,s)a(s)f\left(\lambda \int_0^1 G(s,r)b(r)g(u(r))dr\right)ds \\ &\le \lambda \frac{t}{1-\sum_{i=1}^{m-2} a_i\xi_i} \int_0^1 (1-s)a(s)f\left(\lambda \int_0^1 G(s,r)b(r)g(u(r))dr\right)ds \\ &\le \lambda \frac{t}{1-\sum_{i=1}^{m-2} a_i\xi_i} \int_0^1 (1-s)a(s)(f_0+\epsilon)\lambda \int_0^1 G(s,r)b(r)g(u(r))drds \\ &\le \lambda \frac{1}{1-\sum_{i=1}^{m-2} a_i\xi_i} \int_0^1 (1-s)a(s)(f_0+\epsilon)H_1 ds \le H_1 = \|u\|. \end{aligned}$$

So, $\|Tu\| \le \|u\|$. If we set $\Omega_1 = \{x \in \mathcal{B} \mid \|x\| < H_1\}$, then

$$\|Tu\| \le \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_1. \tag{3.3}$$

Next, from the definitions of f_∞ and g_∞, there exists $\overline{H}_2 > 0$ such that

$$f(x) \ge (f_\infty - \epsilon)x \text{ and } g(x) \ge (g_\infty - \epsilon)x, \quad x \ge \overline{H}_2.$$

Let $H_2 = \max\left\{2H_1, \frac{\overline{H}_2}{\gamma}\right\}$. and $u \in \mathcal{P}$ with $\|u\| = H_2$. Then,

$$\min_{t\in[\xi_{m-2},1]} u(t) \ge \gamma\|u\| \ge \overline{H}_2.$$

As v satisfies the assumptions of Lemmas 1 and 2 by (2.5) and the choice of ϵ, we have for all $s \in [\xi_{m-2}, 1]$

$$\begin{aligned} v(s) &\ge \gamma\|v\| \ge \gamma v(1) \\ &\ge \lambda \frac{\gamma}{1-\sum_{i=1}^{m-2} a_i\xi_i} \sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^1 \xi_i(1-r)b(r)g(u(r))dr \\ &\ge \lambda \frac{\gamma}{1-\sum_{i=1}^{m-2} a_i\xi_i} \sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^1 \xi_i(1-r)b(r)(g_\infty-\epsilon)u(r)dr \\ &\ge \lambda \frac{\gamma^2}{1-\sum_{i=1}^{m-2} a_i\xi_i} \sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^1 \xi_i(1-r)b(r)(g_\infty-\epsilon)dr\|u\| \ge \|u\| = H_2, \end{aligned}$$

and so, from (2.5) and the choice of ϵ, we take

$$Tu(1) \ge \lambda \frac{1}{1-\sum_{i=1}^{m-2} a_i\xi_i} \sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^1 \xi_i(1-s)a(s)f$$

$$\times \left(\lambda \int_0^1 G(s,r)b(r)g(u(r))dr \right) ds$$
$$\geq \lambda \frac{1}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^1 \xi_i (1-s) a(s)(f_\infty - \epsilon)\lambda$$
$$\times \int_0^1 G(s,r)b(r)g(u(r))drds$$
$$\geq \lambda \frac{1}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^1 \xi_i (1-s) a(s)(f_\infty - \epsilon) H_2 ds$$
$$\geq \lambda \frac{\gamma^2}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^1 \xi_i (1-s) a(s)(f_\infty - \epsilon) H_2 ds \geq H_2 = \|u\|.$$

Hence, $\|Tu\| \geq \|u\|$. So, if we set $\Omega_2 = \{x \in \mathcal{B} \mid \|x\| < H_2\}$, then

$$\|Tu\| \geq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{3.4}$$

Applying Theorem 1 to (3.3) and (3.4), we obtain that T has a fixed point $u \in \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$. As such, and with v defined by

$$v(t) = \lambda \int_0^1 G(t,s)b(s)g(u(s))ds,$$

the pair (u, v) is a desired solution of (1.1)–(1.2) for the given λ. The proof is complete. □

Prior to presenting our next result, we define positive numbers L_3 and L_4 by

$$L_3 := \frac{1 - \sum_{i=1}^{m-2} a_i \xi_i}{\gamma} \max \Big\{ \Big[\sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^1 \xi_i (1-r) a(r) f_0 dr \Big]^{-1},$$
$$\Big[\sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^1 \xi_i (1-r) b(r) g_0 dr \Big]^{-1} \Big\},$$

and

$$L_4 := \Big(1 - \sum_{i=1}^{m-2} a_i \xi_i\Big) \min \Big\{ \Big[\int_0^1 (1-r) a(r) f_\infty dr \Big]^{-1},$$
$$\Big[\int_0^1 (1-r) b(r) g_\infty dr \Big]^{-1} \Big\}.$$

Theorem 3. *Assume conditions* (A)–(C) *are satisfied. Then, for each* λ *satisfying*

$$L_3 < \lambda < L_4, \tag{3.5}$$

there exists a pair (u, v) *satisfying* (1.1)–(1.2) *such that* $u(t) > 0$ *and* $v(t) > 0$ *on* $(0, 1)$.

Proof. Let λ be as in (3.5) and $\epsilon > 0$ be chosen such that

$$\frac{1-\sum_{i=1}^{m-2} a_i\xi_i}{\gamma} \max\left\{ \left[\sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^1 \xi_i(1-r)a(r)(f_0-\epsilon)dr\right]^{-1}, \left[\sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^1 \xi_i(1-r)b(r)(g_0-\epsilon)dr\right]^{-1}\right\} \le \lambda$$

and

$$\lambda \le \Big(1-\sum_{i=1}^{m-2} a_i\xi_i\Big) \min\left\{ \left[\int_0^1 (1-r)a(r)(f_\infty+\epsilon)dr\right]^{-1}, \left[\int_0^1 (1-r)b(r)(g_\infty+\epsilon)dr\right]^{-1}\right\}.$$

Let T be the cone preserving, completely continuous operator that was defined by (3.2). From the definitions of f_0 and g_0, there exists $\overline{H_3} > 0$ such that

$$f(x) \ge (f_0-\epsilon)x \text{ and } g(x) \ge (g_0-\epsilon)x, \quad 0 < x \le \overline{H_3}.$$

Also, from the definition of g_0 and the continuity of g it follows that $g(0) = 0$ and so there exists an H_3 with $0 < H_3 < \overline{H_3}$ such that

$$\lambda g(x) \le \frac{\overline{H_3}\left(1-\sum_{i=1}^{m-2} a_i\xi_i\right)}{\int_0^1 (1-r)b(r)dr}, \quad 0 \le x \le H_3.$$

Choose $u \in \mathcal{P}$ with $\|u\| = H_3$. Then, in view of (2.3) for any $s \in [0, 1]$ we have

$$\begin{aligned} v(s) = \lambda \int_0^1 G(s,r)b(r)g(u(r))dr &\le \lambda \frac{s}{1-\sum_{i=1}^{m-2} a_i\xi_i} \int_0^1 (1-r)b(r)g(u(r))dr \\ &\le \lambda \frac{1}{1-\sum_{i=1}^{m-2} a_i\xi_i} \int_0^1 (1-r)b(r)g(u(r))dr \\ &\le \frac{\frac{1}{1-\sum_{i=1}^{m-2} a_i\xi_i}\int_0^1 (1-r)b(r)\overline{H_3}dr}{\frac{1}{1-\sum_{i=1}^{m-2} a_i\xi_i}\int_0^1 (1-r)b(r)dr} = \overline{H_3}. \end{aligned}$$

Hence, in view of Lemmas 1 and 2 for v in place of u and $y(t) = \lambda b(t)g(v(t))$ we have

$$
\begin{aligned}
Tu(1) &\geq \lambda \frac{1}{1-\sum\limits_{i=1}^{m-2} a_i\xi_i} \sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^{1} \xi_i(1-s)a(s) f\Big(\lambda \int_0^1 G(s,r)b(r)g(u(r))dr\Big) \\
&\geq \lambda \frac{1}{1-\sum\limits_{i=1}^{m-2} a_i\xi_i} \sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^{1} \xi_i(1-s)a(s)\times \\
&\qquad \times (f_0-\epsilon)\lambda \frac{1}{1-\sum\limits_{i=1}^{m-2} a_i\xi_i} \sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^{1} \xi_i(1-r)b(r)g(u(r))drds \\
&\geq \lambda \frac{1}{1-\sum\limits_{i=1}^{m-2} a_i\xi_i} \sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^{1} \xi_i(1-s)a(s)\times \\
&\qquad \times (f_0-\epsilon)\lambda \frac{\gamma}{1-\sum\limits_{i=1}^{m-2} a_i\xi_i} \sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^{1} \xi_i(1-r)b(r)(g_0-\epsilon)\|u\|drds \\
&\geq \lambda \frac{1}{1-\sum\limits_{i=1}^{m-2} a_i\xi_i} \sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^{1} \xi_i(1-s)a(s)(f_0-\epsilon)\|u\|ds \\
&\geq \lambda \frac{\gamma}{1-\sum\limits_{i=1}^{m-2} a_i\xi_i} \sum_{i=1}^{m-2} a_i \int_{\xi_{m-2}}^{1} \xi_i(1-s)a(s)(f_0-\epsilon)\|u\|ds \geq \|u\|.
\end{aligned}
$$

So, $\|Tu\| \geq \|u\|$. If we put $\Omega_3 = \{x \in \mathcal{B} \mid \|x\| < H_3\}$, then

$$\|Tu\| \geq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_3. \tag{3.6}$$

Next, by definitions of f_∞ and g_∞, there exists $\overline{H_4}$ such that

$$f(x) \leq (f_\infty + \epsilon)x \text{ and } g(x) \leq (g_\infty + \epsilon)x, \quad x \geq \overline{H_4}.$$

Clearly, since g_∞ is assumed to be a positive real number, it follows that g is unbounded at ∞ and so, there exists $\widetilde{H_4} > \max\{2H_3, \overline{H_4}\}$ such that $g(x) \leq g(\widetilde{H_4})$, for $0 < x \leq \widetilde{H_4}$. Set

$$f^*(t) = \sup_{0\leq s\leq t} f(s), \quad g^*(t) = \sup_{0\leq s\leq t} g(s), \qquad \text{for} \quad t \geq 0.$$

Clearly f^*, g^* are nondecreasing real valued functions for which it holds

$$\lim_{x\to\infty} \frac{f^*(x)}{x} = f_\infty, \quad \lim_{x\to\infty} \frac{g^*(x)}{x} = g_\infty.$$

Hence, there exists $H_4 \geq \overline{H_4}$ such that

$$f^*(x) \leq f^*(H_4), \quad g^*(x) \leq g^*(H_4), \quad \text{for } 0 < x \leq H_4.$$

Choosing $u \in \mathcal{P}$ with $\|u\| = H_4$, by (2.4) we have

$$\begin{aligned}
Tu(t) &= \lambda \int_0^1 G(t,s)a(s)f\Big(\lambda \int_0^1 G(s,r)b(r)g(u(r))dr\Big)ds \\
&\leq \lambda \frac{t}{1-\sum\limits_{i=1}^{m-2} a_i\xi_i} \int_0^1 (1-s)a(s)f^*\Big(\lambda \int_0^1 G(s,r)b(r)g(u(r))dr\Big)ds \\
&\leq \lambda \frac{1}{1-\sum\limits_{i=1}^{m-2} a_i\xi_i} \int_0^1 (1-s)a(s)f^*\Big(\lambda \frac{1}{1-\sum\limits_{i=1}^{m-2} a_i\xi_i} \int_0^1 (1-r)b(r)g^*(u(r))dr\Big)ds \\
&\leq \lambda \frac{1}{1-\sum\limits_{i=1}^{m-2} a_i\xi_i} \int_0^1 (1-s)a(s)f^*\Big(\lambda \frac{1}{1-\sum\limits_{i=1}^{m-2} a_i\xi_i} \int_0^1 (1-r)b(r)g^*(H_4)dr\Big)ds \\
&\leq \lambda \frac{1}{1-\sum\limits_{i=1}^{m-2} a_i\xi_i} \int_0^1 (1-s)a(s)f^*\Big(\lambda \frac{1}{1-\sum\limits_{i=1}^{m-2} a_i\xi_i} \int_0^1 (1-r)b(r)(g_\infty+\epsilon)H_4 dr\Big)ds \\
&\leq \lambda \frac{1}{1-\sum\limits_{i=1}^{m-2} a_i\xi_i} \int_0^1 (1-s)a(s)f^*(H_4)ds \\
&\leq \lambda \frac{1}{1-\sum\limits_{i=1}^{m-2} a_i\xi_i} \int_0^1 (1-s)a(s)(f_\infty+\epsilon)H_4\, ds \leq H_4 = \|u\|,
\end{aligned}$$

and so $\|Tu\| \leq \|u\|$. For this case, if we let $\Omega_4 = \{x \in \mathcal{B} \mid \|x\| < H_4\}$, then

$$\|Tu\| \leq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_4. \tag{3.7}$$

In either of the cases, application of part (ii) of Theorem 1 yields a fixed point u of T belonging to $\mathcal{P} \cap (\overline{\Omega}_4 \setminus \Omega_3)$, which in turn yields a pair (u,v) satisfying (1.1)–(1.2) for the chosen value of λ. The proof is complete. □

4 Discussion

In this section we discuss briefly some other types of m-point boundary value problems. A natural generalization of the m-point boundary value problem (1.1)–(1.2) is system (1.1) with the following boundary conditions ([19]):

$$\begin{aligned}
u(0) = 0, \quad \int_\alpha^\beta h(t)u(t) = u(1), \\
v(0) = 0, \quad \int_\alpha^\beta h(t)v(t) = v(1),
\end{aligned} \tag{4.1}$$

where $[\alpha,\beta]\subset(0,1)$, $h\in C([\alpha,\beta],[0,\infty))$, $\int_\alpha^\beta h(t)tdt\neq 1$ and $\beta\int_\alpha^\beta h(t)dt<1$. By using the same method we can prove eigenvalue results for the system (1.1)–(4.1).

Similar results to that of problem (1.1)–(1.2) can be obtained for other types of m-point boundary value problems, as for example for the following system of m-point boundary value problems for the system of equation (1.1) with the following boundary conditions ([15]):

$$\begin{aligned} u(0)=0,\quad u'(1)=\sum_{i=1}^{m-2}a_iu'(\xi_i),\\ v(0)=0,\quad v'(1)=\sum_{i=1}^{m-2}a_iv'(\xi_i), \end{aligned}\tag{4.2}$$

or, for the following more general m-point boundary value problem (1.1) with the following boundary conditions ([21]):

$$\begin{aligned} u'(0)=\sum_{i=1}^{m-2}a_iu'(\xi_i),\quad u(1)=\sum_{i=1}^{m-2}a_iu(\xi_i),\\ v'(0)=\sum_{i=1}^{m-2}a_iv'(\xi_i),\quad v(1)=\sum_{i=1}^{m-2}a_iv(\xi_i), \end{aligned}\tag{4.3}$$

where $a_i>0$, $i=1,2,\ldots,m-2$ and $0<\xi_1<\ldots<\xi_{m-2}<1$.

For the interested reader we cite below some key relations concerning the problems (1.1)–(4.2) and (1.1)–(4.3).

The Green's function for (1.1)–(4.2) is given by

$$G(t,s)=\begin{cases} s+\dfrac{\sum_{i=1}^{m-1}a_i}{1-\sum_{i=1}^{m-1}a_i}t,\ 0\le t\le 1,\ \xi_{\omega-1}\le s\le\min\{\xi_\omega,t\};\\ \dfrac{1-\sum_{i=\omega}^{m-2}a_i}{1-\sum_{i=1}^{m-2}a_i}t,\ 0\le t\le 1,\max\{\xi_{\omega-1},t\}\le t\le\xi_\omega, \end{cases}$$

where $\omega=1,\ldots,m-1$, while for (1.1)–(4.3) it was proved in [21] that if $\left(1-\sum_{i=1}^{m-2}b_i\right)\left(1-\sum_{i=1}^{m-2}a_i\right)\neq 0$, then for any $y\in C[0,1]$ the following boundary value problem

$$\begin{gathered} u''(t)+y(t)=0,\quad 0<t<1,\\ u'(0)=\sum_{i=1}^{m-2}b_iu'(\xi_i),\quad u(1)=\sum_{i=1}^{m-2}a_iu(\xi_i), \end{gathered}$$

has a unique solution

$$u(t)=-\int_0^1(t-s)y(s)ds+\frac{t}{\sum_{i=1}^{m-2}b_i-1}\sum_{i=1}^{m-2}b_i\int_0^{\xi_i}y(s)ds$$

$$+\frac{1}{1-\sum\limits_{i=1}^{m-2}a_i}\Big(\int_0^1(1-s)y(s)ds-\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}(\xi_i-s)y(s)ds$$

$$-\frac{1}{\sum\limits_{i=1}^{m-2}b_i-1}\sum_{i=1}^{m-2}b_i\int_0^{\xi_i}y(s)ds\Big(1-\sum_{i=1}^{m-2}a_i\xi_i\Big)\Big),$$

for which it holds

$$\inf_{t\in[0,1]}u(t)\geq\gamma\|u\|,$$

where $\gamma=\sum\limits_{i=1}^{m-2}a_i(1-\xi_i)/\Big(1-\sum\limits_{i=1}^{m-2}a_i\xi_i\Big)$, and moreover

$$u(t)\leq\frac{1}{1-\sum\limits_{i=1}^{m-2}a_i}\Big(\int_0^1(1-s)a(s)y(s)ds+\frac{\sum\limits_{i=1}^{m-2}a_i\int_0^{\xi_i}a(s)y(s)ds}{1-\sum\limits_{i=1}^{m-2}a_i}\Big(1-\sum_{i=1}^{m-2}a_i\xi_i\Big)\Big)$$

and

$$y(0)\geq\sum_{i=1}^{m-2}a_i(1-\xi_i)\int_0^1 a(s)y(s)ds.$$

Using these relations and the necessary modifications we can prove similar results to Theorems 2 and 3 for the system of m-point boundary value problems (1.1)–(4.3).

5 Conclusions

The present paper is motivated by both, theoretical interest as well as wide variety of applications in physics and applied mathematics. During the last decades existence of eigenvalues yielding positive solutions for nonlinear second order multi-point boundary value problems is in the focus of interest of many researchers. See, for example [1, 5, 6, 7, 9, 12, 13, 14, 27]. In particular, existence of positive solutions for systems of second order multi-point boundary value problems was studied in [10, 11, 18, 26, 28].

The interest in this paper was focused on the existence of the multi-point boundary value problem (1.1)–(1.2). The main results of the paper are Theorems 2 and 3. Under some rather common conditions on the coefficients a and b and the nonlinear functions f and g, these theorems establish intervals of admissible eigenvalues which yield positive solutions to the boundary value problem (1.1)–(1.2). It turns out that these intervals are determined by the behaviour of the functions f and g at 0 and ∞, by the integrals of the functions $(1-r)\,a\,(r)$ and $(1-r)\,b\,(r)$ on $[0,1]$ and $[\xi_{m-2},1]$ as well as the boundary conditions. The main tool in the technique developed is a fixed point theorem in cones for operators leaving an annular-like region in a Banach space invariant. The appropriate integral operator needed is defined by the use of a

Green function for the corresponding second order scalar equation. As briefly discussed in the previous section, it turns out that the technique developed in the present work can also be used to give existence results for boundary value problems consisting of systems of second order differential equations along with a great variety of boundary value conditions such as (1.1)–(4.1). Furthermore, it will be the subject of a future work to extend and generalize the results of the present work to systems concerning more general type of equations along more general boundary conditions.

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References

[1] R.P. Agarwal, D. O'Regan and P.J.Y. Wong. *Positive Solutions of Differential, Difference and Integral Equations*. Kluwer, Dordrecht, 1999.

[2] M. Benchohra, S. Hamani, J. Henderson, S. K. Ntouyas and A. Ouahab. Positive solutions for systems of nonlinear eigenvalue problems. *Global J. Math. Anal.*, **1**:19–28, 2007.

[3] R. Čiegis, A. Štikonas, O. Štikonienė and O. Subočc. Stationary problems with nonlocal boundary conditions. *Math. Model. Anal.*, **6**(2):178–191, 2001.

[4] R. Čiegis, A. Štikonas, O. Štikonienė and O. Suboč. Monotone finite-difference scheme for parabolical problem with nonlocal boundary conditions. *Differential Equations*, **38**(7):1027–1037, 2002.

[5] L.H. Erbe and H. Wang. On the existence of positive solutions of ordinary differential equations. *Proc. Amer. Math. Soc.*, **120**:743–748, 1994.

[6] J.R. Graef and B. Yang. Boundary value problems for second order nonlinear ordinary differential equations. *Comm. Appl. Anal.*, **6**:273–288, 2002.

[7] D. Guo and V. Lakshmikantham. *Nonlinear Problems in Abstract Cones*. Academic Press, Orlando, 1988.

[8] J. Henderson and S.K. Ntouyas. Positive solutions for systems of nonlinear boundary value problems. *Nonlinear Studies*, **15**:51–60, 2008.

[9] J. Henderson and H. Wang. Positive solutions for nonlinear eigenvalue problems. *J. Math. Anal. Appl.*, **208**:1051–1060, 1997.

[10] J. Henderson and H. Wang. Nonlinear eigenvalue problems for quasilinear systems. *Comput. Math. Appl.*, **49**:1941–1949, 2005.

[11] J. Henderson and H. Wang. An eigenvalue problem for quasilinear systems. *Rocky Mountain J. Math.*, **37**:215–228, 2007.

[12] L. Hu and L. L. Wang. Multiple positive solutions of boundary value problems for systems of nonlinear second order differential equations. *J. Math. Anal. Appl.*, **335**:1052–1060, 2007.

[13] G. Infante. Eigenvalues of some nonlocal boundary value problems. *Proc. Edinburgh Math. Soc.*, **46**:75–86, 2003.

[14] G. Infante and J. R. L. Webb. Loss of positivity in a nonlinear scalar heat equation. *Nonlin. Differ. Equ. Appl.*, **13**:249–261, 2006.

[15] W. Jiang and Y. Guo. Multiple positive solutions for second order m-point boundary value problems. *J. Math. Anal. Appl.*, **327**:415–424, 2007.

[16] R. Liang, J. Peng and J. Shen. Positive solutions to a generalized second order three-point boundary value problem. *Appl. Math. Comput.*, 2007. doi:10.1016/j.amc.2007.07.025.

[17] X. Liu, J. Qiu and Y. Guo. Three positive solutions for second order m-point boundary value problems. *Appl. Math. Comput.*, **156**:733–742, 2004.

[18] R. Ma. Multiple nonnegative solutions of second order systems of boundary value problems. *Nonlinear Anal.*, **42**:1003–1010, 2000.

[19] R. Ma. Positive solutions for second order functional differential equations. *Dynam. Systems Appl.*, **40**:215–223, 2001.

[20] R. Ma. Positive solutions of a nonlinear m-point boundary value problem. *Comput. Math. Appl.*, **42**:755–765, 2001.

[21] R. Ma and N. Castaneda. Existence of solutions of nonlinear m-point boundary value problems. *J. Math. Anal. Appl.*, **256**:556–567, 2001.

[22] S. Pečiulytė and A. Štikonas. On positive eigenfunctions of Sturm-Liouville problem with nonlocal two-point boundary condition. *Math. Model. Anal.*, **12**(2):215–226, 2007.

[23] Y. Raffoul. Positive solutions of three-point nonlinear second order boundary value problems. *Electron. J. Qual. Theory Differ. Equ.*, **15**:1–11, 2002.

[24] M. Sapagovas and R. Čiegis. On some boundary value problems with nonlocal conditions. *Differ. Equations*, **23**(7):1268–1274, 1987.

[25] Y. Sun. Positive solutions of nonlinear second order m-point boundary value problems. *Nonlinear Anal.*, **61**:1283–1294, 2005.

[26] H. Wang. On the number of positive solutions of nonlinear systems. *J. Math. Anal. Appl.*, **281**:287–306, 2003.

[27] J.R.L. Webb. Positive solutions of some three point boundary value problems via fixed point index theory. *Nonlinear Anal.*, **47**:4319–4332, 2001.

[28] Y. Zhou and Y. Xu. Positive solutions of three-point boundary value problems for systems of nonlinear second order ordinary differential equations. *J. Math. Anal. Appl.*, **320**:578–590, 2006.