

FINITE ELEMENT SOLUTION OF BOUNDARY VALUE PROBLEMS WITH NONLOCAL JUMP CONDITIONS*

M. KOLEVA

University of Rousse "Angel Kanchev"

Studentska 8, 7017 Rousse, Bulgaria

E-mail: mkoleva@ru.acad.bg

Received February 4, 2008; revised April 14, 2008; published online September 9, 2008

Abstract. We consider stationary linear problems on non-connected layers with distinct material properties. Well posedness and the maximum principle (MP) for the differential problems are proved. A version of the finite element method (FEM) is used for discretization of the continuous problems. Also, the MP and convergence for the discrete solutions are established. An efficient algorithm for solution of the FEM algebraic equations is proposed. Numerical experiments for linear and nonlinear problems are discussed.

Key words: finite element method, nonlocal jump conditions, well posedness, maximum principle, convergence.

1 Introduction and Problem Formulation

We consider one-dimensional (1D) problem in the interval $r_1 \leq x \leq r_2$, where $r_1 < 0$ and $r_2 > 0$. The interval (r_1, r_2) is divided into three non-overlapping subintervals: $\Omega_1 \equiv (r_1, -t)$, $\Omega_L \equiv (-t, t)$ and $\Omega_2 \equiv (t, r_2)$. A second order differential equation is considered in each of the region

$$-(k_1(x)u_1')' + s_1(x)u_1 = f_1(x), \quad \text{in } \Omega_1, \quad (1.1)$$

$$-(k_2(x)u_2')' + s_2(x)u_2 = f_2(x), \quad \text{in } \Omega_2, \quad (1.2)$$

$$-k_L u_L'' + s_L u_L = f_L(x), \quad \text{in } \Omega_L, \quad (1.3)$$

where

$$k_i(x) \geq k_{i\min} > 0, \quad s_i(x) \geq s_{i\min} > 0, \quad i = 1, 2$$

* This research is supported by the Bulgarian National Fund of Science under Project VU-MI-106/2005

and k_L , s_L are nonnegative constants. The following external boundary conditions are also imposed:

$$u_1(r_1) = \varphi_1, \quad u_2(r_2) = \varphi_2, \quad (1.4)$$

where φ_1, φ_2 are given constants.

The layer Ω_L may have a *structural role* (as in the case of glue), a *thermal role* (as in the case of a thin thermal insulator), an *electromagnetic* or *optical role*, depending on the application. On the two ends of the layer one can impose different, physical possible interface (jump), relations and in some sense independent jump conditions. All these conditions could be classified in four groups:

- **Perfect contact (PC):** $[u]_\Gamma = 0, [w]_\Gamma = 0;$
- **Non-perfect contact (NPC):** $w|_\Gamma = \mu[u]_\Gamma, [w]_\Gamma = 0;$
- **Outer concentrated source (OCS):** $[u]_\Gamma = 0, [w]_\Gamma = \nu;$
- **Own source (OS):** $[u]_\Gamma = 0, [w]_\Gamma = \mu u|_\Gamma,$

where u is the solution of the given differential problem, $w = ku'$ is the flux, μ and ν are given constants.

The interface problems are objects of intensive investigations and numerical methods construction during the past years, see [1, 2, 3, 4, 5, 9, 10, 11, 12, 13, 14, 18] and references given there. In [1], the solution of a general interface problem is reduced to the solution of simpler interface problems of type (PC), (OCS). Conservative difference schemes are studied in [2, 10], while the immersed interface method is developed in [12, 14].

Let us consider the case (PC). Then on $\Gamma = \{-t, t\}$, we have

$$[u]_{(-t)} = u_L(-t) - u_1(-t) = 0, \quad [u]_{(t)} = u_2(t) - u_L(t) = 0, \quad (1.5)$$

$$[ku']_{(-t)} = 0, \quad [ku']_{(t)} = 0. \quad (1.6)$$

The condition (1.5) enforces the continuity of the primary variable u (e.g., temperature), whereas the conditions in (1.6) requires the continuity of the flux $w = ku'$.

Traditionally, there are two ways of handling such layers in the numerical modelling: either they are fully modelled or they are totally ignored. We use the idea of D. Givoli [9] (see also [6]) to replace the layer in which the process is well known, by a fictitious interface, namely a point in 1D case. Special jump conditions are imposed on this interface to model the effect of the layer.

In the layer the problem can be solved analytically. Suppose that $f_L(x)$ is continuous function. Then, the general solution in the layer is of the form

$$u_L(x) = AF(x) + BG(x), \quad (1.7)$$

where F and G are known functions, A and B are unknown constants. Using (1.7), we can write conditions (PC)–(OS) in the following general form

$$k_1(-t)u'_1(-t) = \tilde{\alpha}u_1(-t) + \tilde{\beta}u_2(t) + \tilde{\lambda}, \quad (1.8)$$

$$k_2(t)u'_2(t) = \tilde{\gamma}u_1(-t) + \tilde{\delta}u_2(t) + \tilde{\rho}. \quad (1.9)$$

In [9], for (PC) the values of six unknown coefficients in (1.8), (1.9) are obtained, which are determined in terms of k_L , $F(t)$, $F(-t)$, $G(t)$ and $G(-t)$:

$$\tilde{\alpha} = \alpha, \quad \tilde{\beta} = \beta, \quad \tilde{\gamma} = \gamma, \quad \tilde{\delta} = \delta, \quad \tilde{\lambda} = 0, \quad \tilde{\rho} = 0,$$

$$\alpha = \frac{k_L}{\Delta} [F'(-t)G(t) - G'(-t)G(-t)], \quad \beta = \frac{k_L}{\Delta} [-F(t)F'(-t) + F(-t)G'(-t)],$$

$$\gamma = \frac{k_L}{\Delta} [F'(t)G(t) - G(-t)G'(t)], \quad \delta = \frac{k_L}{\Delta} [F(-t)G'(t) - F(t)F'(t)],$$

$$\Delta = F(-t)G(t) - F(t)G(-t).$$

Thus the equations

$$k_1(-t)u_1'(-t) = \alpha u_1(-t) + \beta u_2(t), \tag{1.10}$$

$$k_2(t)u_2'(t) = \gamma u_1(-t) + \delta u_2(t), \tag{1.11}$$

are equivalent to the original conditions (1.5), (1.6) in the case of (PC).

In the case of (NPC) we obtain

$$\tilde{\alpha} = \frac{1}{D} \left[\alpha + \frac{\alpha\delta}{\mu} + \frac{\gamma^2}{\mu} \right], \quad \tilde{\beta} = \frac{1}{D} \left[\beta + \frac{\beta\delta}{\mu} + \frac{\gamma\delta}{\mu} \right], \quad \tilde{\gamma} = \frac{1}{D} \left[\gamma - \frac{\alpha\gamma}{\mu} - \frac{\alpha\beta}{\mu} \right],$$

$$\tilde{\delta} = \frac{1}{D} \left[\delta - \frac{\alpha\delta}{\mu} - \frac{\beta^2}{\mu} \right], \quad \tilde{\lambda} = 0, \quad \tilde{\rho} = 0, \quad D = \left(1 - \frac{\alpha}{\mu} \right) \left(1 + \frac{\delta}{\mu} \right) + \frac{\beta\gamma}{\mu^2}.$$

In the case (OCS) we derive for the coefficients of the equations (1.8), (1.9):

$$\tilde{\alpha} = \alpha, \quad \tilde{\beta} = \beta, \quad \tilde{\gamma} = \gamma, \quad \tilde{\delta} = \delta, \quad \tilde{\lambda} = -\nu, \quad \tilde{\rho} = \nu,$$

and in the case (OS) we have:

$$\tilde{\alpha} = (\alpha - \mu), \quad \tilde{\beta} = \beta, \quad \tilde{\gamma} = \gamma, \quad \tilde{\delta} = (\delta + \mu), \quad \tilde{\lambda} = 0, \quad \tilde{\rho} = 0.$$

In this paper we shall concentrate on the conditions (PC). The other cases could be treated in a similar way. Thus, we shall solve numerically the equations (1.1), (1.2), subjected to the boundary conditions (1.4) and Robin's type interface conditions (1.10), (1.11).

The remaining part of this paper is organized as follows. In Section 2 we discuss the differential problem. In Section 3 the FEM solution is obtained. In Section 4 we present an efficient approach for solving the generated algebraic equation. The stability and correctness of the proposed algorithm are proved. Also, the FEM solution is analyzed: validation of the discrete MP and the convergence of the discrete solution are established. Numerical results are given in Section 5.

2 Well Posedness of the Differential Problem

In this section, we investigate the well posedness of the problem (1.1), (1.2), (1.4), (1.10), (1.11). Let assume, that for $i = 1, 2$, the data satisfy the usual regularity and ellipticity conditions:

$$k_i(x), s_i(x) \in L_\infty(\Omega_i), \tag{2.1}$$

$$0 \leq k_{0i} \leq k_i(x), \quad 0 \leq s_i(x) \text{ in } \Omega_i, \quad i = 1, 2 \tag{2.2}$$

and the sign conditions:

$$\alpha < 0, \quad \beta > 0, \quad \gamma < 0, \quad \delta > 0. \tag{2.3}$$

Assuming that conditions (2.3) hold, let us consider the special product space $L = L_2(\Omega_1) \times L_2(\Omega_2)$, endowed with the inner product and associated norm

$$(u, v)_L = -\gamma(u_1, v_1)_{L_2(\Omega_1)} + \beta(u_2, v_2)_{L_2(\Omega_2)}, \quad \|v\|_L = (v, v)_L^{1/2},$$

where $(u_i, v_i)_{L_2(\Omega_i)} = \int_{\Omega_i} u_i v_i dx, \quad i = 1, 2$. We can identify $v \in L$ with a scalar function in $\Omega = \Omega_1 \cup \Omega_2$, by $v : \Omega \rightarrow R, v|_{\Omega_i} = v_i, i = 1, 2$. We introduce the product space

$$H^1 = \{v = (v_1, v_2) \mid v_i \in H^1(\Omega_i) \text{ and } v_1(r_1) = 0, v_2(r_2) = 0\},$$

endowed with the inner product

$$(u, v)_{H^1} = -\gamma \left[(u_1, v_1)_{L_2(\Omega_1)} + \left(\frac{du_1}{dx}, \frac{dv_1}{dx} \right)_{L_2(\Omega_1)} \right] + \beta \left[(u_2, v_2)_{L_2(\Omega_2)} + \left(\frac{du_2}{dx}, \frac{dv_2}{dx} \right)_{L_2(\Omega_2)} \right]$$

and the associated norm. We also use the energy inner product and norm

$$[u, v] = -\gamma[u_1, v_1]_1 + \beta[u_2, v_2]_2, \quad |[v]| = [v, v]^{1/2},$$

where

$$[u_i, v_i]_i = \int_{\Omega_i} \left(k_i \frac{du_i}{dx} \frac{dv_i}{dx} + s_i u_i v_i \right) dx, \quad i = 1, 2.$$

First, we derive the weak form of the problem, consisting of (1.1), (1.2), (1.4) and (1.10), (1.11). In Ω_1 the weak form of the equation (1.1) is to find $u_1(x) \in H^1(\Omega_1)$, such that

$$[u_1, v_1] - \alpha u_1(-t)v_1(-t) - \beta u_2(t)v_1(t) = d_1(f_1, v_1), \quad \forall v_1 \in H^1(\Omega_1). \tag{2.4}$$

Analogically, for $u_2(x) \in H^1(\Omega_2)$ we obtain

$$[u_2, v_2] + \gamma u_1(-t)v_2(t) + \delta u_2(t)v_2(t) = d_2(f_2, v_2), \quad \forall v_2 \in H^1(\Omega_2), \tag{2.5}$$

where $d_i(f_i, v) = \int_{\Omega_i} f_i v dx, i = 1, 2$.

To obtain a *symmetric formulation*, we multiply (2.4) by $(-\gamma)$ and (2.5) by β , and add the obtained equalities:

$$\mathcal{A}(u, v) = d(f, v) \quad \forall v \in H^1, \tag{2.6}$$

where

$$\mathcal{A}(u, v) \equiv [u, v] + \mathcal{Z}(u, v), \tag{2.7}$$

$$\mathcal{Z}(u, v) \equiv \gamma\alpha u_1(-t)v_1(-t) + \beta\delta u_2(t)v_2(t) + \beta\gamma[u_1(-t)v_2(t) + u_2(t)v_1(-t)],$$

$$d(f, v) = -\gamma d_1(f_1, v_1) + \beta d_2(f_2, v_2).$$

This leads to the symmetric finite element matrix problem, investigated in Section 3. Thus, under appropriate regularity conditions, the classical problem (1.1), (1.2), (1.4), (1.8), (1.9) is equivalent to the variational problem (2.6).

We state the following important properties of the spaces H^1 and L .

- H^1 and L are Hilbert spaces,
- H^1 is compactly embedded in L .

In the following lemma we deal with some properties of the bilinear form $\mathcal{A}(u, v)$.

Lemma 1. *Under the conditions (2.1)–(2.3) and*

$$\beta\gamma \leq \alpha\delta \tag{2.8}$$

the bilinear form $\mathcal{A}(u, v)$, defined by (2.6), (2.7), is symmetric and bounded on $H^1 \times H^1$. Moreover, this form is also coercive, i.e. there exist a constant $c_0 > 0$ such that

$$\mathcal{A}(v, v) \geq c_0 \|v\|_{H^1}^2, \quad \forall v \in H^1.$$

Proof. The symmetry of \mathcal{A} is obvious, while its boundedness follows from (2.8) and the imbeddings $H^1(\Omega_i) \subset C(\overline{\Omega}_i)$, $i = 1, 2$. Under condition (2.8) we have

$$\mathcal{Z}(v, v) = \alpha\gamma v_1^2(-t) + \beta\delta v_2^2(t) + 2\beta\gamma v_1(-t)v_2(t) \geq 0,$$

which together with (2.2) and the Friedrichs type inequality

$$\int_{\Omega_1} v_1^2(x) dx \leq \frac{(t+r_1)^2}{2} \int_{\Omega_1} \left(\frac{dv_1}{dx}\right)^2 dx \tag{2.9}$$

and similar one for $v_2(x)$ ensures the coerciveness of \mathcal{A} . \square

Theorem 1. *If $f_i \in L_2(\Omega_i)$, $\varphi_i \geq 0$, $i = 1, 2$, and conditions (2.3), (2.8) are satisfied, then the problem (1.1), (1.2), (1.4), (1.10), (1.11) has the unique weak solution $u \in H^1(\Omega_1) \times H^1(\Omega_2)$. If $k_i \in C^1(\overline{\Omega}_i)$, $s_i \in C^1(\overline{\Omega}_i)$, $f_i \in C(\overline{\Omega}_i)$, $i = 1, 2$, then the weak solution belongs to the space $C^2(\overline{\Omega}_1) \times C^2(\overline{\Omega}_2)$ and is the unique classical solution of the problem (1.1), (1.2), (1.4), (1.10), (1.11), which satisfies the following (MP): if $f_i \leq 0$, $i = 1, 2$, then*

$$\max_{i=1,2} \left(\max_{\overline{\Omega}_i} u_i\right) \leq \max\{0, \varphi_1, \varphi_2\}. \tag{2.10}$$

Proof. Lemma 1 allows us to recast the problem (2.6) into the general theory of abstract bilinear forms in Hilbert spaces, see [15]. This ensures the *existence* and *uniqueness* of the weak solution. The proof that the weak solution is a classical solution follows from [15, p. 232], for the classical two-point boundary value problems and we omit it.

We follow the theory in [8]. Let $M := \max\{\varphi_1, \varphi_2\}$ and introduce the piecewise C^1 functions $w_i := \max\{u_i - M, 0\}$, $i = 1, 2$. Then we have $w_i \geq 0$ and $w_1(r_1) = 0$, $w_2(r_2) = 0$. Further, we have $u_i(x) = w_i(x) + M$ for any $x \in \Omega_i$ unless $w_i(x) = 0$. Hence, for this $w = (w_1, w_2)$, the left hand side of (2.6) satisfies

$$\begin{aligned} & -\gamma \int_{r_1}^{-t} [k_1(w_1')^2 + s_1(w_1 + M)w_1] dx + \beta \int_t^{r_2} [k_2(w_2')^2 + s_2(w_2 + M)w_2] dx \\ & + \alpha\gamma(w_1(-t) + M)w_1(-t) + \beta\delta(w_2(t) + M)w_2(t) \\ & + \beta\gamma[(w_1(-t) + M)w_2(t) + (w_2(t) + M)w_1(-t)] \geq 0. \end{aligned}$$

The non-negativity of the first two addends is obvious. The non-negativity of the remained terms of the sum follows from (2.8). The assumption $f_i \leq 0$, $i = 1, 2$ implies that for this w , the right hand side of (2.6) satisfies

$$-\gamma \int_{r_1}^{-t} f_1 w_1 + \beta \int_t^{r_2} f_2 w_2 \leq 0,$$

hence, altogether we have

$$\int_{r_1}^{-t} [k_1(w_1')^2 + s_1(w_1 + M)w_1] dx = 0, \quad \int_t^{r_2} [k_2(w_2')^2 + s_2(w_2 + M)w_2] dx = 0.$$

Therefore $w_1' = 0$ and $w_2' = 0$, i.e. w_i , $i = 1, 2$ are constants. We have seen that $w_1(r_1) = 0$ and $w_2(r_2) = 0$, hence we obtain $w_i = 0$, $i = 1, 2$, which means that (2.10) holds. \square

Remark 1. If $\varphi_i \geq 0$, $i = 1, 2$, then $\max(\max_{\overline{\Omega_1}} u_1, \max_{\overline{\Omega_2}} u_2) = \max(\varphi_1, \varphi_2)$, and if $\varphi_i \leq 0$, $i = 1, 2$, we have the non-positivity property

$$\max_{\overline{\Omega_i}} u_i \leq 0, \quad i = 1, 2.$$

3 Finite Element Method

We consider the uniform partition of the domains Ω_i , $i = 1, 2$:

$$\Omega_1^h = \{x_i = r_1 + (i - 1)h_1, i = 1, \dots, M; h_1 = \frac{-t - r_1}{M - 1}, x_1 = r_1, x_M = -t\},$$

$$\Omega_2^h = \{x_{i+M} = t + (i-1)h_2, i = 1, \dots, N; h_2 = \frac{r_2 - t}{N-1}, x_{M+1} = t, x_{M+N} = r_2\}.$$

Thus the domain $\Omega^h \equiv \Omega_1^h \cup \Omega_2^h$ consists of $M+N-2$ elements $K_i = [x_i, x_{i+1}]$, $i = 1, 2, \dots, M-1, M+1, M+2, \dots, M+N-2$. Every element K_i is assigned a polynomial degree p , $p = 1, 2, 3$. The corresponding standard finite element space of piecewise-polynomial functions $V_{hp} \subset V = V_1 \cup V_2$ has the form

$$V_{hp} = \{v_{hp} \in V; v_{hp} \in P^p(K_i), i = 1, 2, \dots, M+N-2\},$$

where $P^p(K_i)$ stands for the space of polynomials of degree p on the element K_i . The problem is to find $u_{hp} \in V_{hp}$ satisfying

$$\mathcal{A}(u_{hp}, v_{hp}) = d(f, v_{hp}), \quad \forall v_{hp} \in V_{hp}. \tag{3.1}$$

The Galerkin FEM approximation leads to the system of algebraic equations

$$AY = F, \quad u_{hp}(x) = \sum_{i=1}^{M+N} y_i \Phi_i(x), \quad f = \sum_{i=1}^{M+N} f(x_i) \Phi_i(x),$$

$\Phi_i(x)$ is the p -order basis of $V_{hp}(x)$, $\Phi_1 = \Phi_{M+N} = 0$. The matrix A has the forms, given in Figures 1-3.

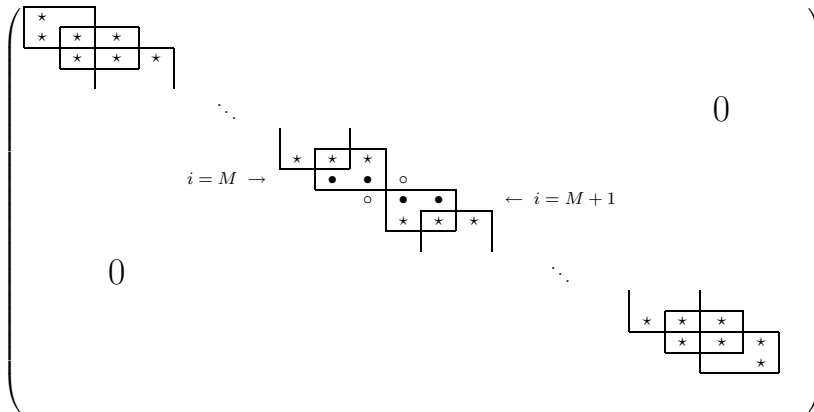


Figure 1. The matrix A of the linear FEM ($p = 1$).

In these figures dots denote coefficients in the equations, corresponding to the interface nodes: x_M and x_{M+1} , black dots (•) denote the coefficients of the unknown solution at grid nodes, which belong to: Ω_1^h for $1 \leq i \leq M$ or Ω_2^h for $M+1 \leq i \leq M+N$, blank dots (◦) denote the coefficients of the unknown solution at grid nodes, which belong to: Ω_2^h for $1 \leq i \leq M$ or Ω_1^h for $M+1 \leq i \leq M+N$.

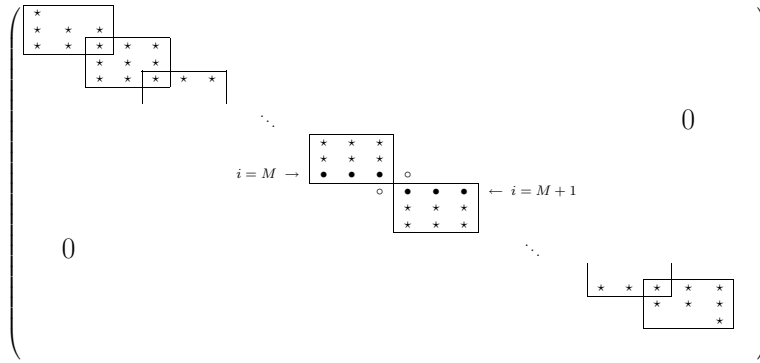


Figure 2. The matrix A of the quadratic FEM ($p = 2$).

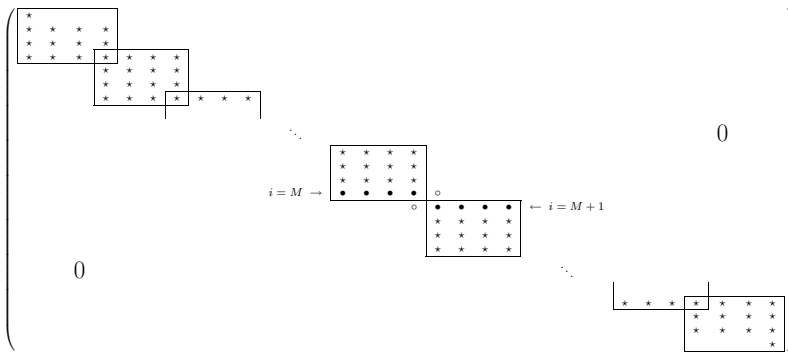


Figure 3. The matrix A of the cubic FEM ($p = 3$).

4 Analysis of the FEM

4.1 Algorithm for solution of the FEM algebraic equations

Numerical experiments show that the error accumulates at interface grid nodes. This way, the idea is to separate the problem into two independent discrete problems and using the right and left Thomas method, to compute the solution at the interface nodes to be determined. We shall show the procedure in the case of *quadratic* FEM. For linear and cubic FEM the same approach is used. The formulas are very long, but standard and they are not given in the present work.

Let us write the system $Ay = F$ in details for quadratic FEM in the case when $k_i, s_i, i = 1, 2$ are constant,

$$f = \begin{cases} -\gamma f_1(x), & x \in \Omega_1^h, \\ \beta f_2(x), & x \in \Omega_2^h, \end{cases} \quad f_i = f(x_i),$$

$$f_{i\pm\frac{1}{2}} = f(x_i \pm h/2), \text{ for } h = \begin{cases} h_1 & \text{in } \Omega_1^h, \\ h_2 & \text{in } \Omega_2^h, \end{cases} \quad f_1 = \varphi_1, \quad f_{M+N} = \varphi_2,$$

$$\downarrow \left\{ \begin{array}{l} y_1 = f_1 \\ A_1 y_1 + B_1 y_{\frac{3}{2}} + C_1 y_2 = \frac{2}{3} h_1 f_{\frac{3}{2}}, \\ \overline{A}_1 y_1 + \overline{B}_1 y_{\frac{3}{2}} + \overline{C}_1 y_2 + \overline{B}_1 y_{\frac{5}{2}} + \overline{A}_1 y_3 = \frac{1}{3} h_1 f_2, \end{array} \right\} \\ \downarrow \left\{ \begin{array}{l} A_1 y_{i-1} + B_1 y_{i-\frac{1}{2}} + C_1 y_i = \frac{2}{3} h_1 f_{i-\frac{1}{2}}, \\ \overline{A}_1 y_{i-1} + \overline{B}_1 y_{i-\frac{1}{2}} + \overline{C}_1 y_i + \overline{B}_1 y_{i+\frac{1}{2}} + \overline{A}_1 y_{i+1} = \frac{1}{3} h_1 f_i, \\ \qquad \qquad \qquad i=3, \dots, M-1 \end{array} \right\} \quad (4.1) \\ \left\{ \begin{array}{l} A_1 y_{M-1} + B_1 y_{M-\frac{1}{2}} - C_1 y_M = \frac{2}{3} h_1 f_{M-\frac{1}{2}}, \\ \overline{A}_1 y_{M-1} + \overline{B}_1 y_{M-\frac{1}{2}} + \overline{C}_1 y_M + \beta \gamma y_{M+1} = \frac{1}{6} h_1 f_M, \end{array} \right\}$$

$$\uparrow \left\{ \begin{array}{l} \beta \gamma y_M + \overline{\overline{C}}_2 y_{M+1} + \overline{B}_2 y_{M+\frac{3}{2}} + \overline{A}_2 y_{M+2} = \frac{1}{6} h_2 f_{M+1} \\ C_2 y_{M+1} + B_2 y_{M+\frac{3}{2}} + A_2 y_{M+2} = \frac{2}{3} h_2 f_{M+\frac{3}{2}}, \end{array} \right\} \\ \uparrow \left\{ \begin{array}{l} \overline{A}_2 y_{i-1} + \overline{B}_2 y_{i-\frac{1}{2}} + \overline{C}_2 y_i + \overline{B}_2 y_{i+\frac{1}{2}} + \overline{A}_2 y_{i+1} = \frac{1}{3} h_2 f_i, \\ C_2 y_i + B_2 y_{i+\frac{1}{2}} + A_2 y_{i+1} = \frac{2}{3} h_2 f_{i+\frac{1}{2}}, \\ \qquad \qquad \qquad i=M+2, \dots, M+N-1 \end{array} \right\} \quad (4.2) \\ \left\{ \begin{array}{l} y_{M+N} = f_{M+N}, \end{array} \right\}$$

where

$$A_1 = \gamma \left(\frac{8k_1}{3h_1} - \frac{2s_1 h_1}{30} \right), \quad B_1 = \gamma \left(-\frac{16k_1}{3h_1} - \frac{16s_1 h_1}{30} \right), \quad C_1 = A_1, \\ \overline{A}_1 = \gamma \left(-\frac{k_1}{3h_1} + \frac{s_1 h_1}{30} \right), \quad \overline{B}_1 = \gamma \left(\frac{8k_1}{3h_1} - \frac{2s_1 h_1}{30} \right), \quad \overline{C}_1 = \gamma \left(-\frac{14k_1}{3h_1} - \frac{8s_1 h_1}{30} \right), \\ A_2 = \beta \left(-\frac{8k_2}{3h_2} + \frac{2s_2 h_2}{30} \right), \quad B_2 = \beta \left(\frac{16k_2}{3h_2} + \frac{16s_2 h_2}{30} \right), \quad C_2 = A_2, \\ \overline{A}_2 = \beta \left(\frac{k_2}{3h_2} - \frac{s_1 2h_2}{30} \right), \quad \overline{B}_2 = \beta \left(-\frac{8k_2}{3h_2} + \frac{2s_2 h_2}{30} \right), \quad \overline{C}_2 = \beta \left(\frac{14k_2}{3h_2} + \frac{8s_2 h_2}{30} \right), \\ \overline{\overline{C}}_1 = \frac{\overline{C}_1}{2} + \gamma \alpha, \quad \overline{\overline{C}}_2 = \frac{\overline{C}_2}{2} + \beta \delta.$$

It is clear that

$$|B_i| \geq |A_i| + |C_i|, \quad i = 1, 2, \quad (4.3)$$

$$|\overline{C}_i| \geq |\overline{A}_i| + |\overline{B}_i|, \quad |\overline{B}_i| > |\overline{A}_i|, \quad i = 1, 2. \quad (4.4)$$

Moreover for small $h_i, i = 1, 2$ we observe that

$$A_i < 0, \quad B_i > 0, \quad C_i < 0, \quad \overline{A}_i > 0, \quad \overline{B}_i < 0, \quad \overline{C}_i > 0, \quad \overline{\overline{C}}_i > 0, \quad i = 1, 2. \quad (4.5)$$

We shall choose the mesh step size, according to the following restrictions:

$$h_i^2 \leq \frac{5k_i}{3s_i}, \quad i = 1, 2. \tag{4.6}$$

Now, taking into account (4.3)–(4.6), we seek the solution in the form

$$\left. \begin{aligned} y_{i-\frac{1}{2}} &= \xi_i^1 y_i + \delta_i^1, \\ y_i &= \xi_i^2 y_{i+\frac{1}{2}} + \eta_i^2 y_{i+1} + \delta_i^2 m, \end{aligned} \right\} \quad i = M - 1, \dots, 2,$$

$$y_{M-\frac{1}{2}} = \xi_M^1 y_M + \delta_M^1,$$

$$y_{M+\frac{3}{2}} = \tilde{\xi}_{M+1}^1 y_{M+1} + \tilde{\delta}_{M+1}^1,$$

$$\left. \begin{aligned} y_i &= \tilde{\xi}_i^2 y_{i-\frac{1}{2}} + \tilde{\eta}_i^2 y_{i-1} + \tilde{\delta}_i^2, \\ y_{i+\frac{1}{2}} &= \tilde{\xi}_i^1 y_i + \tilde{\delta}_i^1, \end{aligned} \right\} \quad i = M + 2, \dots, M + N - 1.$$

The unknown coefficients are calculated as follows:

$$\xi_1^2 = 0, \quad \eta_1^2 = 0, \quad \delta_1^2 = f_1 = \varphi_1,$$

$$\xi_i^1 = \frac{-C_1 - A_1 \eta_{i-1}^2}{\Delta_{i-1}^1}, \quad \delta_i^1 = \frac{\frac{2}{3} h_1 f_{i-\frac{1}{2}} - A_1 \delta_{i-1}^2}{\Delta_{i-1}^1},$$

$$\Delta_{i-1}^1 = A_1 \xi_{i-1}^2 + B_1, \quad i = 2, \dots, M;$$

$$\xi_i^2 = \frac{-\overline{B}_1}{\Delta_{i-1}^2}, \quad \eta_i^2 = \frac{-\overline{A}_1}{\Delta_{i-1}^2}, \quad \delta_i^2 = \frac{\frac{1}{3} h_1 f_i - \overline{A}_1 \delta_{i-1}^2 - \delta_i^1 (\overline{A}_1 \xi_{i-1}^2 + \overline{B}_1)}{\Delta_{i-1}^2},$$

$$\Delta_{i-1}^2 = \xi_i^1 (\overline{A}_1 \xi_{i-1}^2 + \overline{B}_1) + \overline{A}_1 \eta_{i-1}^2 + \overline{C}_1, \quad i = 2, \dots, M - 1;$$

$$\tilde{\xi}_{M+N}^2 = 0, \quad \tilde{\eta}_{M+N}^2 = 0, \quad \tilde{\delta}_{M+N}^2 = f_{M+N} = \varphi_2,$$

$$\tilde{\xi}_i^1 = \frac{-C_2 - A_2 \tilde{\eta}_{i+1}^2}{\tilde{\Delta}_{i+1}^1}, \quad \tilde{\delta}_i^1 = \frac{\frac{2}{3} h_2 f_{i+\frac{1}{2}} - A_2 \tilde{\delta}_{i+1}^2}{\tilde{\Delta}_{i+1}^1},$$

$$\tilde{\Delta}_{i+1}^1 = A_2 \tilde{\xi}_{i+1}^2 + B_2 \quad i = M + N - 1, \dots, M + 1;$$

$$\tilde{\xi}_i^2 = \frac{-\overline{B}_2}{\tilde{\Delta}_{i+1}^2}, \quad \tilde{\eta}_i^2 = \frac{-\overline{A}_2}{\tilde{\Delta}_{i+1}^2}, \quad \tilde{\delta}_i^2 = \frac{\frac{1}{3} h_2 f_i - \overline{A}_2 \tilde{\delta}_{i+1}^2 - \tilde{\delta}_i^1 (\overline{A}_2 \tilde{\xi}_{i+1}^2 + \overline{B}_2)}{\tilde{\Delta}_{i+1}^2},$$

$$\tilde{\Delta}_{i+1}^2 = \tilde{\xi}_i^1 (\overline{A}_2 \tilde{\xi}_{i+1}^2 + \overline{B}_2) + \overline{A}_2 \tilde{\eta}_{i+1}^2 + \overline{C}_2, \quad i = M + N - 1, \dots, M + 2.$$

Therefore, if we substitute $y_{M-\frac{1}{2}} = \xi_M^1 y_M + \delta_M^1$, $y_{M-1} = \xi_{M-1}^2 (\xi_M^1 y_M + \delta_M^1) + \eta_{M-1}^2 y_M + \delta_{M-1}^2$ in (4.1) and $y_{M+\frac{3}{2}} = \tilde{\xi}_{M+1}^1 y_{M+1} + \tilde{\delta}_{M+1}^1$, $y_{M+2} = \tilde{\xi}_{M+2}^2 (\tilde{\xi}_{M+1}^1 y_{M+1} + \tilde{\delta}_{M+1}^1) + \tilde{\eta}_{M+2}^2 y_{M+1} + \tilde{\delta}_{M+2}^2$ in (4.2), we obtain

$$\begin{bmatrix} y_M \\ y_{M+1} \end{bmatrix} = H^{-1} \begin{bmatrix} \frac{1}{6} h_1 f_M - \delta_M^1 (\overline{B}_1 + \overline{A}_1 \xi_{M-1}^2) - \overline{A}_1 \delta_{M-1}^2 \\ \frac{1}{6} h_2 f_{M+1} - \tilde{\delta}_{M+1}^1 (\overline{B}_2 + \overline{A}_2 \tilde{\xi}_{M+2}^2) - \overline{A}_2 \tilde{\delta}_{M+2}^2 \end{bmatrix},$$

where

$$H = \begin{bmatrix} \xi_M^1 (\overline{B}_1 + \overline{A}_1 \xi_{M-1}^2) + \overline{A}_1 \eta_{M-1}^2 + \overline{C}_1 & \beta\gamma \\ \beta\gamma & \tilde{\xi}_{M+1}^1 (\overline{B}_2 + \overline{A}_2 \tilde{\xi}_{M+2}^2) + \overline{A}_2 \tilde{\eta}_{M+2}^2 + \overline{C}_2 \end{bmatrix}.$$

The existence of the inverse matrix H^{-1} is ensured by the conditions (2.8), (4.4)–(4.6).

Although A is not a diagonal dominant matrix, which is a classical requirement, see [16, 17], for the correctness and stability of the Thomas method, we can prove the following statement.

Theorem 2. *The algorithm described above is correct and stable.*

Proof. For linear FEM solution, the details are given in [13]. For quadratic FEM, using (4.3)–(4.6), just as in the classical theory, [16, 17] and applying the full induction method, we can prove the inequalities

$$\begin{aligned} |\xi_i^1| \leq 1, \quad \xi_i^1 > 0, \quad \Delta_{i-1}^1 > 0, \quad i = 2, \dots, M, \\ |\xi_i^2| \leq 1, \quad \xi_i^2 > 0, \quad |\eta_i^2| \leq 1, \quad \eta_i^2 < 0, \quad \Delta_{i-1}^2 > 0, \quad i = 2, \dots, M - 1, \end{aligned} \tag{4.7}$$

$$0 < \xi_i^1 \xi_{i-1}^2 + \eta_{i-1}^2 < 1, \quad i = 2, \dots, M - 1, \tag{4.8}$$

$$\begin{aligned} |\tilde{\xi}_i^1| \leq 1, \quad \tilde{\xi}_i^1 > 0, \quad \tilde{\Delta}_{i+1}^1 > 0, \quad i = M + N - 1, \dots, M + 1; \\ |\tilde{\xi}_i^2| \leq 1, \quad \tilde{\xi}_i^2 > 0, \quad |\tilde{\eta}_i^2| \leq 1, \quad \tilde{\eta}_i^2 < 0, \quad \tilde{\Delta}_{i+1}^2 > 0, \end{aligned} \tag{4.9}$$

$$0 < \tilde{\xi}_i^1 \tilde{\xi}_{i+1}^2 + \tilde{\eta}_{i+1}^2 < 1, \quad i = M + N - 1, \dots, M + 2. \tag{4.10}$$

Correctness. We shall note that existence of H^{-1} is provided from the assertion: $\det(H) > 0$. Indeed, using the coefficients of the solution of the left and right Thomas method, (2.8) and (4.6), we have

$$\det(H) = \left(-\frac{\overline{B_1}}{\xi_M^2} - \frac{\overline{C_1}}{2} + \alpha\gamma \right) \left(-\frac{\overline{B_2}}{\xi_M^2} - \frac{\overline{C_2}}{2} + \beta\delta \right) - \beta^2\gamma^2 > 0.$$

Now, taking into account the inequalities for Δ in (4.7) and (4.9) we prove that the algorithm is correct.

Stability. We have to show that computing the solution, the error does not increase. Suppose that there has been a mistake ε , for some i . Thus, for example, from the right Thomas method we have

$$\begin{aligned} y_{i-\frac{1}{2}} &= \xi_1^1(y_i + \varepsilon) + \delta_i^1, \\ y_{i-1} &= \xi_{i-1}^2 y_{i-\frac{1}{2}} + \eta_{i-1}^2(y_i + \varepsilon) + \delta_{i-1}^2 \\ &= \xi_{i-1}^2 [\xi_1^1(y_i + \varepsilon) + \delta_i^1] + \eta_{i-1}^2(y_i + \varepsilon) + \delta_{i-1}^2 \\ &= (\xi_i^1 \xi_{i-1}^2 + \eta_{i-1}^2) y_i + (\xi_i^1 \xi_{i-1}^2 + \eta_{i-1}^2) \varepsilon + \xi_{i-1}^2 \delta_i^1 + \delta_{i-1}^2. \end{aligned}$$

Now, the stability of the algorithm follows from (4.7)–(4.10).

Finally, we note that the same approach and argumentation can be applied for the cubic FEM. \square

Now, the problem can be solved separately in domains Ω_1 and Ω_2 . Numerical experiments show that in the case of **one layer**, there is no essential difference in the results, using FEM on the whole interval Ω or computing the solution in Ω_1 and Ω_2 separately, applying the above approach. The effect (computational time and accuracy) of such separation into independent problems comes when the computational domains are very large or/and in the case of multilayer domain.

4.2 Discrete MP and Convergence

The preservation of characteristic qualitative properties of different phenomena becomes a more and more important requirement to the construction of reliable numerical models, [7]. In this section we discuss the discrete version of the continuous maximum principle given in Theorem 1.

DEFINITION 1. We say that the problem (3.1) satisfies the discrete maximum principle if for any right-hand side $f \in L^2(\Omega)$ and $\varphi_i \geq 0, i = 1, 2$ it holds

$$f \geq 0 \text{ in } \Omega \Rightarrow u_{hp} \geq 0 \text{ in } \Omega.$$

Theorem 3. *If the requirements of Theorem 1 and Lemma 1 are fulfilled, then the problem (3.1) satisfies the discrete maximum principle and the following estimate is true for all f and $\varphi_i, i = 1, 2$*

$$\|y\|_C \leq \overline{M} \|f\|_C, \tag{4.11}$$

where

$$\overline{M} = \max \left\{ 1 - \frac{(r_1 - t)^2}{24\gamma k_1}, 1 - \frac{(r_2 - t)^2}{24\gamma k_2} \right\}, \quad \|z\|_C = \max_{\overline{\Omega}^h} |z|.$$

The error at the mesh points satisfies

$$\|u(x_i) - y(x_i)\|_C \leq Ch^4, \quad i = 1, \dots, M + N, \tag{4.12}$$

where the constant C is independent of $h, h = \max\{h_1, h_2\}$.

Proof. The proof is achieved taking advantage of the classical approach, see [16, 17], to analyze the Thomas method solution by means of the corresponding coefficients. We shall discuss in details the solution, obtained by quadratic FEM. The same analysis can be applied for the linear and cubic FEM solutions.

First, as before, using the full induction method, (4.3)–(4.5), (4.7)–(4.10) and the obvious inequality $\overline{B}_i A_i \geq \overline{A}_i B_i$, we prove (for h arbitrary small):

$$\delta_i^1 > 0, \quad i = 2, \dots, M \quad \text{and} \quad \tilde{\delta}_i^1 > 0, \quad i = M + N - 1, \dots, M + 1, \tag{4.13}$$

$$\delta_i^2 > 0, \quad i = 2, \dots, M \quad \text{and} \quad \tilde{\delta}_i^2 > 0, \quad i = M + N - 1, \dots, M + 1. \tag{4.14}$$

Next, we observe that $y_M > 0$ and $y_{M+1} > 0$. Actually, the matrix H is diagonally dominant and the sign condition (positive elements in the main diagonal and negative elements in upper and lower diagonal) is satisfied, owing to (4.3)–(4.6) and (2.8). Moreover, the right hand side of the system for y_M and y_{M+1} is positive, taking into account that $f_i > 0, i = 1, \dots, M + N$ and (4.13), (4.14).

First, we use consequently the formulas of the solution of the right Thomas method, starting with $y_{M-\frac{1}{2}}$. From (4.7) and (4.13), it is evidently that $y_{M-\frac{1}{2}} > 0$. Next, for y_{M-1} we have

$$\begin{aligned} y_{M-1} &= \xi_{M-1}^2 y_{M-\frac{1}{2}} + \eta_{M-1}^2 y_M + \delta_M^2 \\ &= \xi_{M-1}^2 (\xi_M^1 y_M + \delta_M^1) + \eta_{M-1}^2 y_M + \delta_M^2 \\ &= (\xi_{M-1}^2 \xi_M^1 + \eta_{M-1}^2) y_M + \xi_{M-1}^2 \delta_M^1 + \delta_M^2. \end{aligned}$$

Thus, (4.7), (4.8), (4.13) and (4.14) implies that $y_{M-1} > 0$. Further, for $y_{M-\frac{3}{2}}$, using (4.7) and (4.13) we obtain $y_{M-\frac{3}{2}} = \xi_{M-1}^1 y_{M-1} + \delta_{M-1}^1 > 0$. Next, in the same manner, we prove for $y_{M-2}, y_{M-\frac{5}{2}}, y_{M-3}, y_{M-\frac{7}{2}}, \dots, y_2, y_{\frac{3}{2}}$ that they all are positive. For the left Thomas method we apply the same arguments.

Next, we substitute the solution at half grid nodes in the solution at integer grid nodes, denote $\zeta_i := \xi_i^2 \xi_{i+1}^1 + \eta_i^2$, $\tilde{\zeta}_i := \xi_i^2 \delta_{i+1}^1 + \delta_i^2$ and for the right Thomas solution we obtain

$$y_i = \zeta_i y_{i+1} + \tilde{\zeta}_i, \quad i = M - 1, \dots, 2 \tag{4.15}$$

$$y_{i-\frac{1}{2}} = \xi_i^1 y_i + \delta_i^1, \quad i = M, \dots, 2, \tag{4.16}$$

at half grid nodes.

Let $y = y^\circ + y^*$, where y° is the solution of the left and right Thomas method for $\varphi_1 = \varphi_2 = 0, f_i \neq 0, i = 2, \dots, M + N - 1$ and y^* is obtained for $f_i = 0, i = 2, \dots, M + N - 1, \varphi_1 \neq 0, \varphi_2 \neq 0$.

For (4.15), from (4.8), we have

$$|y_i| \leq |\zeta_i| |y_{i+1}| + |\tilde{\zeta}_i| \leq |y_{i+1}| + |\tilde{\zeta}_i|$$

and hence

$$|y_i| \leq \sum_{k=i}^{M-1} |\tilde{\zeta}_k|, \quad 2 \leq i \leq M - 1, \tag{4.17}$$

where

$$|\tilde{\zeta}_i| \leq |\xi_i^2| |\delta_{i+1}^1| + |\delta_i^2| \leq |\delta_{i+1}^1| + |\delta_i^2|.$$

First we estimate y° . For $\delta_{i+1}^1 = \frac{1}{A_1 \xi_i^2 + B_1} \left(\frac{2}{3} h_1 f_{i+\frac{1}{2}} - A_1 \delta_i^2 \right)$ we find

$$\begin{aligned} |\delta_{i+1}^1| &\leq \frac{1}{|B_1| - |A_1|} \left(\left| \frac{2}{3} h_1 f_{i+\frac{1}{2}} \right| + |A_1 \delta_i^2| \right) \\ &\leq \frac{1}{|B_1| - |A_1|} \left(\left| \frac{2}{3} h_1 f_{i+\frac{1}{2}} \right| + \left| \frac{A_1}{A_1 \xi_{i-1}^2 + B_1} \right| \left| \frac{2}{3} h_1 f_{i-\frac{1}{2}} + A_1 \delta_{i-1}^2 \right| \right) \\ &\leq \frac{1}{|B_1| - |A_1|} \left(\left| \frac{2}{3} h_1 f_{i+\frac{1}{2}} \right| + \left| \frac{2}{3} h_1 f_{i-\frac{1}{2}} + A_1 \delta_{i-1}^2 \right| \right) \leq \dots \\ &\leq \frac{2h_1}{3(|B_1| - |A_1|)} \sum_{k=1}^i |f_{k+\frac{1}{2}}|. \end{aligned} \tag{4.18}$$

Next, substituting δ_i^1 in δ_i^2 , we have

$$\delta_i^2 = \frac{\frac{1}{3} h_1 f_i - \bar{A}_1 \delta_{i-1}^2 + \left(\frac{2}{3} h_1 f_{i-\frac{1}{2}} - A_1 \delta_{i-1}^2 \right) \frac{-\bar{A}_1 \xi_{i-1}^2 - \bar{B}_1}{A_1 \xi_{i-1}^2 + B_1}}{\xi_i^1 (\bar{A}_1 \xi_{i-1}^2 + \bar{B}_1) + \bar{A}_1 \eta_{i-1}^2 + \bar{C}_1}.$$

From (4.3), (4.4), (4.7), (4.9) and the obvious relations $|\overline{B}_i| > B_i$, $\overline{A}_i > |A_i|$, $|\overline{B}_i| > \overline{A}_i$, $B_i > |A_i|$ it follows that

$$|\delta_i^2| \leq \frac{1}{|\overline{C}_1| - |\overline{B}_1|} \left| \frac{1}{3} h_1 f_i + \frac{2}{3} h_1 f_{i-\frac{1}{2}} \right| + \left| \frac{A_1}{\xi_i^1 (\overline{A}_1 \xi_{i-1}^2 + \overline{B}_1) + \overline{A}_1 \eta_{i-1}^2 + \overline{C}_1} \right| |\delta_{i-1}^2|$$

$$\leq \frac{1}{|\overline{C}_1| - |\overline{B}_1|} \left(\left| \frac{1}{3} h_1 f_i \right| + \left| \frac{2}{3} h_1 f_{i-\frac{1}{2}} \right| \right) + |\delta_{i-1}^2|.$$

Consequently, we obtain

$$|\delta_i^2| \leq \frac{h_1}{3(|\overline{C}_1| - |\overline{B}_1|)} \left(\sum_{k=1}^{i-1} |f_{k+1}| + 2 \sum_{k=1}^{i-1} |f_{k+\frac{1}{2}}| \right). \tag{4.19}$$

Solution of (4.16) leads to the similar estimate with $\delta_i^2 = 0$.

Substituting (4.18) and (4.19) in (4.17) we arrive at the next inequality

$$\|y^\circ\|_C \leq \frac{h_1}{3(|\overline{C}_1| - |\overline{B}_1|)} \sum_{i=2}^{M-1} \left(\sum_{k=1}^{i-1} |f_{k+1}| + 4 \sum_{k=1}^{i-1} |f_{k+\frac{1}{2}}| + 2|f_{i+\frac{1}{2}}| \right). \tag{4.20}$$

Similarly, for y^* we obtain

$$\|y^*\|_C \leq |\varphi_1|. \tag{4.21}$$

Summing (4.20) and (4.21) we get the estimate for the solution of the right Thomas method

$$\|y\|_C \leq |\varphi_1| + \frac{h_1}{3(|\overline{C}_1| - |\overline{B}_1|)} \sum_{i=2}^{M-1} \left(\sum_{k=1}^{i-1} |f_{k+1}| + 4 \sum_{k=1}^i |f_{k+\frac{1}{2}}| \right). \tag{4.22}$$

The same arguments can be applied for the left Thomas method solution to obtain

$$\|y\|_C \leq |\varphi_2| + \frac{h_2}{3(|\overline{C}_2| - |\overline{B}_2|)} \sum_{i=M+2}^{M+N-1} \left(\sum_{k=1}^{i-1} |f_{k+1}| + 4 \sum_{k=1}^i |f_{k+\frac{1}{2}}| \right). \tag{4.23}$$

Combining (4.22) and (4.23) we obtain

$$\|y\|_C \leq \max\{|\varphi_1|, |\varphi_2|\} + \frac{1}{3} \max \left\{ \frac{h_1}{|\overline{C}_1| - |\overline{B}_1|} \sum_{i=2}^{M-1} \left(\sum_{k=1}^{i-1} |f_{k+1}| + 4 \sum_{k=1}^i |f_{k+\frac{1}{2}}| \right), \right.$$

$$\left. \frac{h_2}{|\overline{C}_2| - |\overline{B}_2|} \sum_{i=M+2}^{M+N-1} \left(\sum_{k=1}^{i-1} |f_{k+1}| + 4 \sum_{k=1}^i |f_{k+\frac{1}{2}}| \right) \right\}$$

and hence we arrive at (4.11). The estimate (4.12) follows straightforward from (4.11). \square

5 Numerical Results

Example 1. (*Linear case*) Let's take the following data:

$$r_1 = -6, \quad t = 1, \quad r_2 = 3.5, \quad f_1(x) = me^x(s_1 - k_1) + s_1e^{-t}, \quad \varphi_1 = e^{-t} + me^{r_1},$$

$$f_2(x) = ne^{-x}(s_2 - k_2) + 2s_2e^{-t}, \quad \varphi_2 = 2e^{-t} + ne^{-r_2}.$$

The exact solution of the problem (1.1),(1.2),(1.4),(1.8),(1.9) with $\alpha = -6$, $\beta = 1$, $\gamma = -1$, $\delta = 2$, $k_1(x) = 4$, $k_2(x) = 2$, $s_1(x) = 4$, $s_2(x) = 2$ is

$$u(x) = e^{-t} + me^x \text{ in } \Omega_1, \quad u(x) = 2e^{-t} + ne^{-x} \text{ in } \Omega_2.$$

The mesh step size is $h_1 = h_2 = h$, $m = -0.4872$, $n = -0.8718$. The computational results (i.e. the errors and the convergence rate) are given in Table 1.

Table 1. Error in the maximal discrete norm E_∞^h and CR (written in brackets).

h	Linear FEM	Quadratic FEM	Cubic FEM
0.1	8.2736e-5	2.7259e-8	6.3498e-9
0.05	2.0682e-5 (2.0001)	1.7780e-9 (3.9384)	4.0458e-10 (3.9722)
0.025	5.1708e-6 (1.9999)	1.1399e-10 (3.9633)	2.5172e-11 (4.0065)
0.0125	1.2928e-6 (1.9999)	8.1375e-12 (3.8082)	1.5709e-12 (4.0022)
0.00625	3.2319e-7 (2.0000)	5.5288e-13 (3.8795)	9.8011e-14 (4.0025)

The convergence rate (CR) is calculated using double mesh principle:

$$CR = \log_2 \frac{E_\infty^h}{E_\infty^{\frac{h}{2}}}, \quad E_\infty^h = \max_{1 \leq i \leq M+N} |u_{hp}(x_i) - u(x_i)|.$$

In Figure 4(a) we have plotted the exact solution and the numerical solution, computed with quadratic FEM. The results of experiments confirm the theoretical estimate (4.12).

Example 2. (*Nonlinear case*) For applications, it is interesting to compute the solution of problem (1.1),(1.2),(1.4),(1.10),(1.11), where the Poisson-Boltzmann equation is solved

$$f_i = K_i \sinh(u_i - \varphi_i), \quad i = 1, 2.$$

In this example, the parameters are:

$$K_1 = K_2 = -1, \quad \varphi_1 = 1, \quad \varphi_2 = 2, \quad k_1 = s_1 = 4,$$

$$k_2 = s_2 = 2, \quad \alpha = -6, \quad \beta = 1, \quad \gamma = -1, \quad \delta = 2.$$

We use the Newton method for solving the obtained nonlinear system. Having in mind the Taylor series of function $\sinh(\cdot)$, the most appropriate initial solution (for starting the Newton iterations) is the solution of the linear problem (1.1),(1.2),(1.4),(1.10),(1.11) with $f_i = K_i(u_i - \varphi_i)$, $i = 1, 2$.

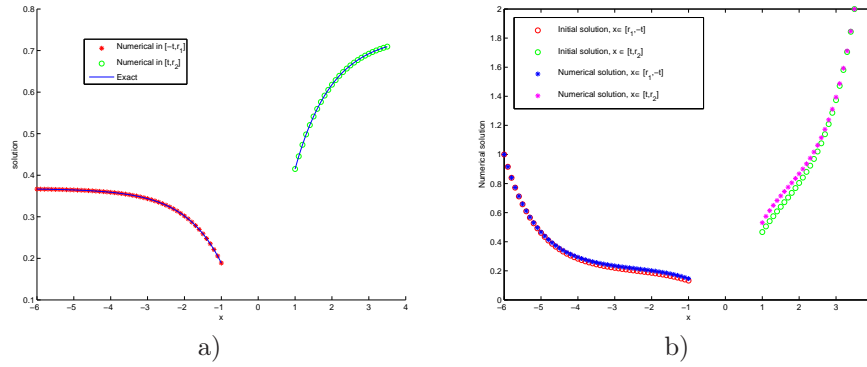


Figure 4. a) Exact and numerical solution, $h = 0.1$, $r_1 = -6$, $t = 1$, $r_2 = 3.5$; b) Initial and numerical solution, $h = 0.1$, $r_1 = -6$, $t = 1$, $r_2 = 3.5$.

Table 2. Error in the maximal discrete norm (E_∞^h) and CR values.

h	Linear FEM	Quadratic FEM	Cubic FEM
0.1	3.51930e-4	1.38425e-7	8.22015e-8
0.05	8.80811e-5 (1.9983)	9.51208e-9 (3.8632)	5.26157e-9 (3.9656)
0.025	2.20262e-5 (1.9996)	6.12831e-10 (3.9562)	3.31664e-10 (3.9877)
0.0125	5.50668e-6 (2.0000)	3.90228e-11 (3.9731)	2.07909e-11 (3.9957)
0.00625	1.37643e-6 (2.0003)	2.46561e-12 (3.9843)	1.29943e-12 (4.0000)
0.003125	3.43848e-7 (2.0011)	1.54229e-13 (3.9988)	8.11861e-14 (4.0005)

The results are presented in Table 2, for $r_1 = -2.5$, $t = 1$, $r_2 = 2$. As an exact solution we take the numerical solution, computed with a very small mesh step $h = 0.0001$, i.e. $M = 15001$ and $N = 10001$.

The experiment shows, that the algorithm can be applied also to solve nonlinear problems and the estimate (4.12) is still valid.

In Figure 4(b) we have plotted the initial solution and the numerical solution, computed with the quadratic FEM, $h = 0.1$, $r_1 = -6$, $t = 1$, $r_2 = 3.5$.

6 Conclusions

Interface problems of the following types have attracted a lot of attention from both theoretical and numerical analysts over the last years:

- The differential equation/system has discontinuous, but bounded coefficients;
- The differential equation/system has singular source term, such as Dirac-delta functions;
- The interface can be fixed or moving with time;
- There are one or several interfaces;
- Problems that are defined on irregular domains.

In all of the cases above, usually the differential problems are defined on joint domains for which the interfaces are internal boundaries.

The present work discusses a type of interface problems defined on disjoint intervals, namely with nonlocal jump conditions. Well posedness and maximum principle for the differential problems are proved. We performed FEM discretizations, based on linear, quadratic and cubic bases. The discrete maximum principle and convergence results for the discrete problems are studied. Also, an economic algorithm for solution of the generated FEM algebraic systems of equations is realized. Correctness and stability of the proposed algorithm are proved. The numerical results for linear and nonlinear test examples give a good agreement with the theoretical ones.

It is interesting to be considered problems in which the nonlocal jump conditions (1.8) – (1.9) are coupled together through some nonlinear relations. Also, it is desirable to develop the FEM for parabolic interface problems on disjoint domains, as well as for elliptic and parabolic two-dimensional problems.

References

- [1] I. Angelova and L. Vulkov. Singularly perturbed differential equations with discontinuous coefficients and concentrated factors. *Appl. Math. & Comp.*, **158**:683–701, 2004.
- [2] M. Buike and A. Buikis. Modelling of three-dimensional transport process in anisotropic layered stratum by conservative averaging method. *WSEAS Transactions of Heat and Mass Transfer*, **1**(4):430–437, 2006.
- [3] R. Čiegis, A. Ilgevičius, H. Liess, M. Meilūnas and O. Suboč. Numerical simulation of the heat conduction in electrical cables. *Math. Model. Anal.*, **12**(4):425–439, 2007.
- [4] R. Čiegis and O. Iliev. Numerical algorithms for modelling of liquid polymer moulding. *Math. Model. Anal.*, **8**(3):181–202, 2003.
- [5] R. Čiegis, A. Papastavrou and A. Zemitis. Additive splitting methods for elliptic-parabolic problems. *Annali del Universiteta di Ferrara- Sez. VII, Sc. Mat.*, **46**:291–306, 2000.
- [6] B. Cook, T. Kazakova, S. Lyu, P. Madrid, J. Neal, M. Pauletti and R. Zhao. Cell-foreign particle interaction. ima preprint series. pp. 2133–3, 2006.
- [7] I. Farago and R. Horvath. On the nonnegativity conservation of finite element solutions of parabolic problems. *CAKUTO Int. Ser. Math. Sci. Appl.*, **15**:76–84, 2001.
- [8] D. Gilbarg and N. Trudinger. *Elliptic Partial Differential Equations of Second Order, 3rd edition*. Springer, 2001.
- [9] D. Givoli. Finite element modeling of thin layers. *CMES*, **5**(6):497–531, 2004.
- [10] B. Jovanovich and L. Vulkov. On the convergence of finite difference schemes for parabolic problems with concentrated capacity. *Numer. Math.*, **89**:715–734, 2001.
- [11] H. Kalis and I. Kangro. Increasing of accuracy for engineering calculation for heat transfer problem in two layer media. *Math. Model. Anal.*, **10**(2):173–170, 2005.

- [12] J. Kandilarov and L. Vulkov. The immersed interface method for two-dimensional heat-diffusion equations with singular own sources. *Appl. Num. Math.*, **57**:486–497, 2007.
- [13] M. Koleva. Finite element solution of 1D boundary value linear and nonlinear problems with nonlocal jump conditions. *AIP Conf. Pr.*, **946**:163–170, 2007.
- [14] Z. Li. and K. Ito. The immersed interface method: Numerical solutions of PDEs involving interfaces and irregular domains. *SIAM Frontiers in Applied Mathematics*, **33**, 2006.
- [15] V. Mikhailov. *Partial differential equations*. Nauka, Moscow, 1980. (in Russian)
- [16] A. A. Samarskii. *The Theory of Difference Schemes*. Marcel Dekker Inc., 2001.
- [17] A. A. Samarskii and E. S. Nikolaev. *Numerical methods for grid equations*. Nauka, Moscow, 1978. (in Russian)
- [18] L. Vulkov. Formulation and analysis of stationary nonlinear problems on non-connected layers with distinct material properties. *AIP Conf. Pr.*, **946**:188–195, 2007.