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THE MAXIMUM PRINCIPLE AND ITS APPLICATION FOR THE ANALYSIS OF DIFFERENCE SCHEMES

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ABSTRACT

The subject of this paper is the maximum principle and its application for the analysis difference schemes. To some extent, it is a survey on construction and investigation of some new classes of monotone difference schemes. The established maximum principle for derivatives has a principal meaning. The coefficient stability of difference schemes in Banach spaces is proved on the base of this principle. New results on unconditional stability of difference schemes with weights, conservative explicit-implicit schemes (staggered schemes) are given.

1. INTRODUCTION

The subject of this paper is the maximum principle and its application in investigations of stability and convergence of difference schemes. It is well known that the difference schemes satisfying the maximum principle are called monotone. In other words, solution of monotone difference schemes is stable in the norm C. It is very important to save the property of monotonicity of difference schemes for numerical solution of applied problems with the aid of a computer.

It is necessary to note a direct closeness of the notions — a monotone scheme and conditionality of the system of equations $Ax = g$, which arise on numerical solution of the corresponding difference scheme. Thus, the difference boundary problem that satisfies the maximum principle is called the well conditional problem. Such definition of well conditionality is equivalent to one of the accepted definitions in the theory of systems of linear equations when they consider the number $\|A\| \cdot \|A^{-1}\|$ as a measure of its conditionality.

To some extent this paper is a survey on construction and investigation of some new classes of monotone difference schemes. In Section 2, the formulation of the maximum principle (more exactly — its consequence) in one-dimensional and multidimensional cases is given. The maximum principle in a very comfortable view on investigation of boundary problems with not uniform boundary conditions is given. It is known that only one scheme with the weight $\sigma = 1$ for parabolic equation is unconditionally stable, i.e., when establishing an a priori estimate of stability in the norm C a connection between the time step τ, spatial steps h_1, h_2,...,h_p, and the coefficients of the equation is absent. In Section 3, it is shown that all the schemes with the weight $\sigma \geq 1$ possess such a property.

In Section 4, the maximum principle is used for investigating new classes of conservative explicit-implicit schemes (staggered schemes). The maximum principle for derivatives established in Section 5 deserves a special attention. In Section 6, the coefficient stability of difference schemes in Banach spaces is proved on the base of this principle.

2. STATEMENT OF THE MAXIMUM PRINCIPLE

The statement of the grid maximum principle one can be found in many textbooks and monographs (see, e.g., [6; 8]). To obtain a priori estimates in the norm C various corollaries are mostly used, which are given below.

2.1. One-dimensional case

Let the function $y_i = y(x_i)$ defined on a uniform grid

$$\overline{\omega}_h = \{x_i = ih,\ i = 0, 1, \dots, N\} = \omega_h \cup \{x_0 = 0;\ x_N = l\}, \qquad (2.1)$$

be the solution of the problem

$$A_i y_{i-1} - C_i y_i + B_i y_{i+1} = -F_i, \quad i = 1, 2, \dots, N-1, \quad y_0 = y_N = 0. \quad (2.2)$$

Let us define the grid norms:

$$\|\cdot\|_C = \max_{x \in \omega_h} |\cdot|, \quad \|\cdot\|_{\overline{C}} = \max_{x \in \overline{\omega}_h} |\cdot|.$$

Lemma 2.1 [6]. *Suppose that*

$$A_i > 0,\ B_i > 0,\ D_i = C_i - A_i - B_i > 0, \quad i = 1, 2, \dots, N-1.$$

Then for the solution of problem (2.2) *the following estimate holds:*

$$\|y\|_{\overline{C}} \leq \|F/D\|_C. \tag{2.3}$$

2.2. Multi-dimensional case

In the rectangle $\overline{G} = \{0 \leq x_\alpha \leq l_\alpha,\ \alpha = 1, 2, \ldots, p\}$ with the boundary Γ we set up a uniform grid

$$\overline{\omega}_h = \omega_h \cup \gamma_h,\ \omega_h = \{x_i = (x_1^{(i_1)}, \ldots, x_p^{(i_p)}),\ i_\alpha = 1, \ldots, N_\alpha - 1,$$
$$x_\alpha^{(0)} = 0,\ x_\alpha^{(N_\alpha)} = l_\alpha,\ \alpha = 1, 2, \ldots, p\}$$

with constant steps $h_1 = x_1^{(i_1)} - x_1^{(i_1-1)}, \ldots, h_p = x_p^{(i_p)} - x_p^{(i_p-1)}$, γ_h is a set of the boundary nodes. To apply the maximum principle in getting the estimates of stability, one should reduce the difference scheme to the following canonical form [8, p. 293]:

$$A(x)y(x) = \sum_{\xi \in S'(x)} B(x,\xi)y(\xi) + F(x), \quad x \in \omega_h, \quad y(x) = \mu(x), \quad x \in \gamma_h, \tag{2.4}$$

and examine the following sufficient conditions on the coefficients:

$$A(x) > 0, \quad B(x,\xi) \geq 0, \quad D(x) = A(x) - \sum_{\xi \in S'(x)} B(x,\xi) > 0, \quad x \in \omega_h. \tag{2.5}$$

Here $S'(x) = S(x) \backslash \{x\}$, $S(x)$ is a stencil of the scheme.

Lemma 2.2 [10]. *Let the positivity property of the coefficients* (2.5) *be satisfied. Then for the solution of the problem* (2.4) *the following estimate is valid:*

$$\|y\|_{\overline{C}} \leq \max\left\{\|y\|_{C_\gamma}, \|F/D\|_C\right\}, \tag{2.6}$$

where $\|\cdot\|_{\overline{C}} = \max\limits_{x \in \omega_h \cup \gamma_h} |\cdot|$, $\|\cdot\|_{C_\gamma} = \max\limits_{x \in \gamma_h} |\cdot|$, $\|\cdot\|_C = \max\limits_{x \in \omega_h} |\cdot|$.

The Lemma given is very suitable to study the stability of difference schemes simultaneously with respect to boundary conditions and right-hand side (especially for nonstationary problems).

3. WEIGHTED SCHEMES

Let us consider the weighted difference scheme

$$y_t = y_{\bar{x}x}^{(\sigma)}, \quad \hat{y}_0 = \hat{y}_N = 0, \quad y(x,0) = u_0(x), \quad x \in \omega_h, \tag{3.1}$$

which approximates the simplest initial-boundary value problem for the parabolic equation. Here we use the standard notation of the theory of difference schemes [6]:

$$y = y_i^j = y(x_i, t_j),\ x_i \in \omega_h,\ t_j \in \omega_\tau = \{t_j = j\tau,\ j = 0, 1, \dots, j_0 - 1,\ t_{j_0} = T\},$$

$$y_t = (\hat{y} - y)/\tau,\ \hat{y} = y_i^{j+1},\ y_{\bar{x}x} = (y_{i+1} - 2y_i + y_{i-1})/h^2,\ y^{(\sigma)} = \sigma\hat{y} + (1-\sigma)y,$$

$0 \le \sigma \le 1$ is a real parameter. Varying it, we can get the schemes with different features: from pure explicit ($\sigma = 0$) to pure implicit ($\sigma = 1$). For solution of this scheme the maximum principle is valid [6]:

$$\max_{t \in \omega_\tau} \|y(t)\|_C \le \|u_0\|_C \tag{3.2}$$

under the condition

$$\tau \le \frac{h^2}{2(1-\sigma)}. \tag{3.3}$$

From the condition (3.3) it follows that if $\sigma = 1$, then the monotone difference scheme (3.1) is unconditionally stable in the norm C (there are no constraints on a ratio of the grid steps τ and h). If $\sigma \ne 1$, then the scheme is monotone under the constraint (3.3) only.

Let us show that for any $\sigma \ge 1$ the scheme (3.3) is also unconditionally monotone. Actually using the identity $y_t = (y^{(\sigma)} - y)/(\sigma\tau)$ the difference scheme (3.1) is reduced to the canonical form (2.2):

$$\frac{\sigma\tau}{h^2} y_{i-1}^{(\sigma)} + \left(1 + \frac{2\sigma\tau}{h^2}\right) y_i^{(\sigma)} + \frac{\sigma\tau}{h^2} y_{i+1}^{(\sigma)} = -y, \quad y_0^{(\sigma)} = y_N^{(\sigma)} = 0.$$

Since all the conditions of Lemma 2.1 are fulfilled, then from inequality (2.3) we get the estimate

$$\|y^{(\sigma)}\|_C \le \|y\|_C.$$

As $\|y^{(\sigma)}\|_C \ge \sigma\|\hat{y}\|_C - (\sigma - 1)\|y\|_C$ provided that $\sigma \ge 1$, we conclude that for any given σ a priori estimate (3.2) is valid for solution of difference scheme (3.1).

Now we consider pure implicit scheme ($\sigma = 1$) with inhomogeneous boundary conditions:

$$y_t = \hat{y}_{\bar{x}x} + \varphi, \quad \hat{y}_0 = \mu_1(\hat{t}), \quad \hat{y}_N = \mu_2(\hat{t}), \quad y(x,0) = u_0(x), \quad x \in \omega_h. \tag{3.4}$$

Rewriting the difference scheme (3.4) in the canonical form (2.4), we see that the positivity property of the coefficients (2.5) is carried out. Further, applying Lemma 2.2, we obtain the following estimate of accuracy in the uniform metric [10]:

$$\max_{t \in \overline{\omega}_\tau} \|y(t)\|_{\overline{C}} \le \max\left\{\|y(0)\|_C,\ \max_{t \in \omega_\tau} \|y(t)\|_{C_\gamma}\right\} + \sum_{t \in \omega_\tau} \tau\|\varphi(t)\|_C.$$

4. EXPLICIT-IMPLICIT SCHEMES

For numerical simulation of problems with singularities it is often convenient to apply hybrid methods. These methods are based on the use of various difference schemes in the corresponding domains.

These algorithms can often be written as schemes with variable weight factors [7; 12]:

$$y_t = (y_{\bar{x}x})^{(\sigma)} + \varphi, \tag{4.1}$$

where $\sigma = \sigma(x,t)$, $(x,t) \in \omega = \omega_h \times \omega_\tau$. In the case of an explicit-implicit scheme

$$\sigma = \sigma_i = \begin{cases} 0, & \text{if } i \text{ is even,} \\ 1, & \text{if } i \text{ is odd.} \end{cases} \tag{4.2}$$

In spite of the implicitness the solution of this scheme is obtained explicitly. It appears that the condition of its stability in the norm C

$$\tau \le h^2 \tag{4.3}$$

allows one to find numerical solution with the time step τ two times bigger in comparison with the familiar explicit scheme ($\sigma = 0$).

Explicit-implicit schemes were popular in 60-70 years. Then they have been almost forgotten. A few years ago simultaneously A. Gulin, A. Samarskii [2], R. Čiegis [1], and P. Matus, A. Lapin, I. Mikhiliouk [3; 4] turned our attention to these schemes again, from absolutely different points of view. To our mind, the main defect is that these schemes are nonconservative in consequence of the failure to carry out the equality

$$(y_{\bar{x}x,i})^{(\sigma_i)} \ne \left((y_{\bar{x}})^{(\sigma_i)}\right)_{x,i},$$

when the weight σ depends on a grid node. Therefore, in the case of these schemes one fails to obtain a priori estimates in more weak norms and to prove convergence of the difference schemes with reduced requirements to the properties of the solution of a differential problem.

4.1. Conservative schemes

To construct a conservative method, let us write a scheme with variable weight factors in the following form (for simplicity we consider homogeneous boundary conditions):

$$y_t = \left(y_{\bar{x}}^{(\sigma)}\right)_x + \varphi, \quad y_i^0 = u_{0i}, \quad y_0^{j+1} = y_N^{j+1} = 0. \tag{4.4}$$

Consider numerical implementation of scheme (4.4) with weight factors (4.2). Let i be even. Then, from equation (4.4) it follows that

$$y_i^{j+1} = y_i^j + \frac{\tau}{h}\left(y_{\bar{x},i+1}^{j+1} - y_{\bar{x},i}^{j}\right) + \tau\varphi_i^j,$$
$$y_{i+1}^{j+1} = y_i^j + \frac{\tau}{h}\left(y_{\bar{x},i+2}^{j} - y_{\bar{x},i+1}^{j+1}\right) + \tau\varphi_{i+1}^j.$$

Hence, it is easy to find y_i^{j+1}, y_{i+1}^{j+1} from the explicit formulas.

4.2. Stability with respect to the initial data and right-hand side

The following statement is valid [3].

Theorem 4.1. *Suppose*

$$0 \le \sigma_i^n \le 1, \quad \frac{\sigma_i^n + \sigma_{i+1}^n}{2} \le 1, \quad \tau \le \frac{h^2}{2 - (\sigma_i^n + \sigma_{i+1}^n)}. \tag{4.5}$$

Then for solution of the difference scheme (4.4) *with arbitrary weight factors* σ_i^n *satisfying the conditions* (4.5) *the following estimate holds:*

$$\max_{t\in\overline{\omega}_\tau} \|y(t)\|_{\overline{C}} \le \|y(0)\|_C + \sum_{t\in\omega_\tau} \tau\|\varphi(t)\|_C. \tag{4.6}$$

Corollary 4.1. Assume that conditions (4.5) are satisfied and $\tau \le h^2$. Then conservative explicit-implicit scheme (4.2), (4.4) is stable in the norm C and a priori estimate (4.6) is correct.

4.3. Multi-dimensional case

In the two-dimensional case the corresponding conservative difference scheme has the form

$$y_t = \left((y_{\bar{x}_1})^{(\sigma_1)}\right)_{x_1} + \left((y_{\bar{x}_2})^{(\sigma_2)}\right)_{x_2} + \varphi,$$

where

$$\sigma_k = \sigma_k(x_k) = \begin{cases} 0, & \text{if } i_k \text{ is even}, \\ 1, & \text{if } i_k \text{ is odd}, \end{cases} \qquad x_k = i_k h_k.$$

The appropriate a priori estimate of stability in the norm C (4.6) is fulfilled provided that $h_1 = h_2 = h$ and

$$\tau \le \frac{h^2}{(2 - (\sigma_1 + \sigma_{1+})) + (2 - (\sigma_2 + \sigma_{2+}))}.$$

5. THE MAXIMUM PRINCIPLE FOR DERIVATIVES

For a one-dimensional parabolic equation consider the implicit scheme

$$y_t = \Lambda\hat{y} + \varphi, \quad \Lambda\hat{y} = \hat{y}_{\bar{x}x}, \tag{5.1}$$

$$y(x,0) = u_0(x), \quad x \in \omega_h, \quad y|_{\gamma_h} = \mu = \text{const.} \tag{5.2}$$

From the identity $\hat{v} = v + \tau v_t$, it follows that equation (5.1) can be rewritten in the equivalent form

$$\hat{v} = \tau\hat{v}_{\bar{x}x} + \tau\varphi_{\bar{t}}, \quad \hat{v}_0 = \hat{v}_N = 0, \tag{5.3}$$

where $v = y_{\bar{t}}$. The problem (5.3) is reduced to the form (2.2), if

$$A_i = B_i = \tau/h_i^2, \quad C_i = 1 + A_i + B_i, \quad D_i = 1.$$

It is obvious that for arbitrary τ and h_i the conditions of Lemma 2.1 are satisfied and the recurrence relation

$$\|y_{\bar{t}}^{j+1}\|_C \le \|y_{\bar{t}}^{j}\|_C + \tau\|\varphi_{\bar{t}}^{j}\|_C \tag{5.4}$$

holds by virtue of the estimate (2.3). Assuming $t = 0$, we rewrite equation (5.1) in the form

$$y_t = \tau y_{t\bar{x}x} + \Lambda u_0 + \varphi(0).$$

Further we apply the maximum principle to the grid function $v = y_t(0)$, $v(0) = v(x_i, 0)$ with $F = \Lambda u_0 + \varphi(0)$. As a result, from the recurrence relation (5.4), we get the estimate

$$\|y_{\bar{t}}\|_C \le \|\Lambda u_0 + \varphi(0)\|_C + \sum_{t'=\tau}^{t} \tau\|\varphi_{\bar{t}}(t')\|_C. \tag{5.5}$$

Next, substituting into (5.5) the value of y_t from equation (5.1) and applying the triangle inequality $\|\Lambda\hat{y} + \varphi\| \ge \|\Lambda\hat{y}\| - \|\varphi\|$, we have

$$\|\Lambda\hat{y}\|_C = \|\hat{y}_{\bar{x}x}\|_C \le \|\Lambda u_0 + \varphi(0)\|_C + \|\varphi\|_C + \sum_{t'=\tau}^{t} \tau\|\varphi_{\bar{t}}(t')\|_C. \tag{5.6}$$

The last estimate and the obvious inequality $\|\hat{y}\|_C \le \|y\|_C + \tau\|y_t\|_C$ yield the following:

$$\|y\|_C \le \|u_0\|_C + t\left(\|\Lambda u_0 + \varphi(0)\|_C + \sum_{t'=\tau}^{t-\tau} \tau\|\varphi_{\bar{t}}(t')\|_C\right). \tag{5.7}$$

Thus we have the following result.

Theorem 5.1. *For the solution of the problem* (5.1), (5.2) *for any* $t \in \omega_\tau$ *a priori estimates* (5.5) – (5.7) *hold.*

Schemes for the multi-dimensional equations are constructed by analogy, namely, we have

$$y_t = \Lambda \hat{y} + \varphi, \quad y_0 = u_0, \tag{5.8}$$

$$\Lambda = \sum_{\alpha=1}^{p} \Lambda_\alpha, \quad \Lambda_\alpha y = (a_\alpha(x) y_{\bar{x}_\alpha})_{x_\alpha}, \quad x \in \omega_h, \quad y|_{\gamma_h} = 0. \tag{5.9}$$

Difference scheme (5.8), (5.9) satisfies the grid maximum principle. Thus, from Lemma 2.2 we have

$$\max_{t\in\omega_\tau} \|y(t)\|_C \le \|y(0)\|_C + \sum_{t'=0}^{t-\tau} \tau \|\varphi(t')\|_C. \tag{5.10}$$

On the other hand, differentiating the difference equation (5.8) in t by analogy with (5.3), we get

$$v_t = \Lambda \hat{v} + \varphi_t.$$

Further arguments lead us to the estimates (5.5), (5.6), where the operator Λ is defined from (5.9).

6. COEFFICIENT STABILITY

By means of the maximum principle for derivatives proved above we shall obtain a priori estimates of stability of the solution when the coefficients are perturbed. Obviously, the problem considered is very important since in mathematical modeling of applied problems the coefficients of equations can be given inaccurately. For instance, they have been computed as a result of experimental observation, determination, etc.

In the nonstationary processes, which are described by parabolic equations with the given boundary and initial conditions

$$\frac{\partial u}{\partial t} = Lu + f(x,t), \quad u(x,t)|_\Gamma = \mu(t), \quad u(x,0) = u_0(x),$$

the variable t (time) plays the special role and that is why we have to distinguish it. Here L is a differential operator that works on $u(x,t)$, where $x = (x_1, x_2, \dots, x_p)$ is a point in a p-dimensional domain G with a bound Γ. The function $u(x,t)$ is an element of the Banach space B for all fixed t. That is why instead of $u(x,t)$ we get an abstract function $u(t)$ of the variable $0 \le t \le t_0$ with a range space in B, i.e., $u(t) \in B$ for all $t \in [0, t_0]$. The operator L is replaced by the operator A given in B. Thus, we obtain an abstract Cauchy problem:

$$\frac{du}{dt} + Au = f(t), \quad 0 < t \leq t_0, \quad u(0) = u_0. \tag{6.1}$$

The Cauchy problem is called stable with respect to the initial data and right-hand side, if

$$\|\tilde{u}(t) - u(t)\| \leq M_1\|\tilde{u}_0 - u_0\| + M_2 \int_0^t \|\tilde{f}(t') - f(t')\|dt', \tag{6.2}$$

where M_1, M_2 are positive constants and $\tilde{u}(t)$ is a solution of the following problem with perturbed entrance data:

$$\frac{d\tilde{u}}{dt} + A\tilde{u} = \tilde{f}(t), \quad 0 < t \leq t_0, \quad \tilde{u}(0) = \tilde{u}_0. \tag{6.3}$$

Let the operator A be constant, linear, and unbounded. Then definition (6.2) is equivalent to the inequality

$$\|u(t)\| \leq M_1\|u_0\| + M_2 \int_0^t \|f(t')\|dt'. \tag{6.4}$$

Estimates like (6.2), (6.4) are well known in the theory of differential equations. However, the coefficients of the equations are an entrance data also. Why then has this problem not been solved so far? Firstly, in view of the relation

$$\tilde{A}\tilde{u} - Au \neq A(\tilde{u} - u),$$

the problem for the perturbation of solution $z(t) = \tilde{u}(t) - u(t)$ has already become nonlinear; secondly, it is not clear how to estimate the norm of unbounded operator. In [5; 9; 11; 12] the coefficient stability both in continuous and discrete cases was proved in Hilbert spaces only.

Consider the problem of coefficient stability in the Banach space. Using the notation given above, let us approximate on the time grid ω_τ the differential problem (6.1) by the difference one

$$y_t + A\hat{y} = \varphi, \quad y_0 = u_0. \tag{6.5}$$

The corresponding perturbed difference scheme has the form

$$\tilde{y}_t + \tilde{A}\hat{\tilde{y}} = \tilde{\varphi}, \quad \tilde{y}_0 = \tilde{u}_0. \tag{6.6}$$

Subtracting from the last expression the previous one, we get the problem for the perturbation $z = \tilde{y} - y$

$$z_t + A\hat{z} = (\tilde{\varphi} - \varphi) - (\tilde{A} - A)\hat{\tilde{y}}. \tag{6.7}$$

Suppose that for the solution of the problem (6.5) the following estimate is valid:

$$\|y\| \le \|y_0\| + \sum_{t'=0}^{t-\tau} \tau\|\varphi\|. \tag{6.8}$$

Then for the problem (6.7) the inequality

$$\|z\| \le \|\tilde{u}_0 - u_0\| + \sum_{t'=0}^{t-\tau} \tau \left\{ \|\tilde{\varphi} - \varphi\| + \|(\tilde{A} - A)\hat{\tilde{y}}\| \right\} \tag{6.9}$$

holds. About the perturbation of the nonuniformly bounded operator A we assume that

$$\|(\tilde{A} - A)\hat{\tilde{y}}\| \le \alpha\|\tilde{A}\hat{\tilde{y}}\|. \tag{6.10}$$

Here $\alpha \ge 0$ is a low bound of a set of constants satisfying (6.10). It specifies a measure of vicinity of in general unbounded operators.

Substituting (6.10) into (6.9), we have

$$\|z\| \le \|\tilde{u}_0 - u_0\| + \sum_{t'=0}^{t-\tau} \tau\|\tilde{\varphi} - \varphi\| + \alpha \sum_{t'=0}^{t-\tau} \tau\|\tilde{A}\hat{\tilde{y}}(t')\|. \tag{6.11}$$

Using the assumption (6.8) and estimation technique (5.6), the last term in (6.11) is estimated in the following way:

$$\alpha \sum_{t'=0}^{t-\tau} \tau\|\tilde{A}\hat{\tilde{y}}(t')\| \le \alpha \sum_{t'=0}^{t-\tau} \tau \left\{ \|\tilde{\varphi}(0) - \tilde{A}\tilde{u}_0\| + \|\tilde{\varphi}\| + \sum_{t''=\tau}^{t'} \tau\|\tilde{\varphi}_{\bar{t}}(t'')\| \right\}. \tag{6.12}$$

Estimates (6.11), (6.12) demonstrate stability with respect to perturbation of the initial data, right-hand side, and operator (strong stability).

As an example, let us consider one-dimensional problem (6.5) with the constant operator

$$Ay = -(ay_{\bar{x}})_x, \quad x \in \omega_h, \quad y_0 = y_N = 0,$$

and the norm $\|\cdot\| = \|\cdot\|_C$. Then

$$(\tilde{A} - A)\tilde{y} = -\left((\tilde{a} - a)\tilde{y}_{\bar{x}}\right)_x.$$

Using the appropriate embedding theorem, it is easy to show that the estimate (6.10) is carried out, where

$$\alpha = \alpha_0 \max\left\{\|\tilde{a} - a\|_C,\ \|\tilde{a}_{\bar{x}} - a_{\bar{x}}\|_C\right\}.$$

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Maksimumo principas ir jo naudojimas baigtinių skirtumų schemų analizėje

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Straipsnis yra apžvalginis. Jame apibendrinti rezultatai, skirti maksimumo principo naudojimui, analizuojant baigtinių skirtumų schemų stabilumą ir konvergavimą. Didžiausias dėmesys skiriamas sprendinio išvestinių įverčiams. Remiantis šiuo maksimumo principo variantu įrodomas kai kurių baigtinių skirtumų schemų koeficientinis stabilumas Banacho erdvėse. Taip pat ištirtos ekonomiškos schemos, skirtos daugiamačių uždavinių sprendimui, įvertintas skaitinio sprendinio tikslumas, kai naudojamas netolygus diskretusis tinklas.